A Comprehensive Method for Specifying Tolerance Requirements for Assemblies

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Charles David Carr, graduate student
Department of Mechanical Engineering
Brigham Young University

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ABSTRACT

When mechanical parts are assembled, their dimensional and form variations accumulate statistically and propagate kinematically, causing critical assembly features to vary. The main objective of this thesis was to relate engineering tolerance requirements to manufacturing constraints by means of a common engineering model for variation in assemblies. A comprehensive system for defining assembly specifications (engineering requirements) and for analytically predicting assembly variations was developed, which will allow closer control of product quality and performance.

Vector loops were used to model mechanical assemblies. The loops join relevant dimensions into chains. Closed loops define the kinematic constraints on the assembly. Open loops define engineering constraints. Linearized equations derived from the vector loops were used to solve for the assembly variations. The linearized equations are expressed in terms of scalar sensitivities and known component tolerances. Simplified methods were developed for determining sensitivities using nodal coordinates.

Eight distinct assembly specifications were identified, patterned after the ANSI Y14.5-M 1982 standards for geometric dimensioning and tolerancing. They are general enough to cover a wide range of assembly applications. For each case, the required form of the loop equations was determined and solution procedures have been demonstrated for computing assembly variations and comparing the results to the specified assembly limits. A new set of engineering drawing symbols has also been proposed which is similar to ANSI Y14.5-M 1982.
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Chapter 1 Introduction

There are two major areas where tolerances play an important part in new product development: 1) Product design and 2) Process design. The design engineer must develop a mechanism or assembly that performs some mechanical function or meets some sort of design criterion. Several parts of an assembly must work together to give the desired results. The design specification often results in a specified overall tolerance of an assembly parameter such as, a gap or an angle between two components. On the process design side each part of the mechanism or assembly is manufactured individually to certain dimensions with specified tolerances.

The design engineer may have specified component tolerances necessary to ensure that the parts fit together, but may be puzzled over why his overall specification isn't met when he gets the machined parts back and assembles them. Parts are often reworked or discarded, and design changes are made for the tolerance of one or more parts only to find out later that the design change didn't work. More tolerances are then tightened on components, and manufacturing costs escalate. Both the design engineer and the manufacturer become frustrated with one another, and they lose confidence in the ability of the other to perform. Proper tolerance analysis is the link that is needed to bring these two areas together.

![Diagram](image)

Figure 1.1 Tolerance Analysis is the link between Product Design and Process Design.

1.1 Motivation

The motivation behind this thesis comes from the clash that often occurs between design and manufacturing personnel over assembly tolerances and component tolerances. Much effort has been put into standardizing how to indicate and check variations on
individual parts [ANSI 1982], but in almost every design application the variation of an
assembly is what is really wanted. Not to belittle the work that has been done in the area
of component variation, assembly variation analysis would not be possible without
component variation standards. This is because assembly variation is a function of the
accumulation variation of the component variations.

Engineering requirements are applied to assemblies. Manufacturing tolerances are
applied to component parts. Assembly tolerance analysis relates the engineering
requirements to the manufacturing constraints.

Tolerances on individual parts are a function of the manufacturing process. Depending
on the specified tolerance, different manufacturing processes may be used to
meet those specifications. Component tolerances can be independently controlled by
changing the manufacturing process and checked with precision gauges.

Assembly variations are a function of the combined tolerances of the components
and how they interact kinematically. Depending on the specified assembly tolerance,
component tolerances may need to be changed to meet those specifications.

Designers can specify tolerances on individual components, and more time and
better equipment (more money) may be the only requirements to meet those specifications.
How can a designer know how all the components and their tolerances will interact
kinematically to see if the assembly specification is met? And if they aren't, then what
tolerances are needed to assure that it is?

There needs to be a way for a designer to analyze an assembly of parts so he can
determine the variation in an assembly before each component is actually manufactured and
the system is put together physically. Using worst case or statistical methods the designer
should be able to know the bounds on the assembly variation or a probability that the
assembly variation will meet the designer's requirements. Giving the designer this
information will relieve tension between designers and manufacturers by allowing the
designer to model his assembly more accurately and let him resolve problems before they
show up in the manufacturing process. Cost will also be lowered by resolving problems in
the design stage, before parts are actually manufactured.

Fortini and Marler have done some work in the area of representing assemblies with
vector chains [Fortini 1967, Marler 1988]. Marler extended Fortini's work by linearizing
vector loop equations and solving for variations in kinematic dimensions using linear
algebra. This will be described more fully in chapter 2.
1.2 CATS

CATS (Computer Aided Tolerance Selection) is a software package being developed by Dr. Kenneth W. Chase at Brigham Young University with the help of several graduate and undergraduate students. It is designed to help designers choose component tolerances and analyze assembly tolerances. The addition of algorithms which will expand the capability of the software to analyze problems that have assembly specifications is very much needed. The existing specifications in CATS were: gaps, lengths, and kinematic lengths and angles. Computer implemented algorithms also speed up the analysis process by eliminating the need for lengthy hand calculations and can eliminate the need to compute coordinates, angles, lengths, and vector loop paths that can be retrieved from a graphical preprocessor.

1.3 Objective

The objective of this thesis is to develop a comprehensive system for specifying assembly tolerance requirements and analyzing the variation of assemblies, which is a result of accumulated variations of several components.

Specifically the objective is to:

1. Show similarities or parallels between component variation and assembly variation.
2. Reduce the need for written notes on drawings to indicate specifications for assembly variations by developing assembly specification symbols.
3. Develop mathematical methods for analyzing a wide variety of assembly variations.
4. Implement the mathematical methods into CATS and enhance its data structure to allow addition of assembly specifications.
Chapter 2. Theory

Assemblies are made up of individually manufactured component parts. Many different manufacturing processes may have been used in producing each of the assembly components. Some parts may have been cast or molded, others may have been turned on a lathe, while others may have been precision ground. With all of the different processes and sizes involved, each part will have different variations or tolerances associated with its dimensions. When the parts are assembled the overall, or resultant, assembly dimensions are formed. Assembly dimensions vary from one assembly to the next as a result of process variations in component part dimensions. Assembly dimensions are thus dependent upon the independently varying manufactured part dimensions.

2.1 Kinematic Variations

When the components are assembled, small adjustments are made between mating parts to accommodate manufacturing variations. We will call these adjustments kinematic variations. Kinematic adjustments occur where mating parts are joined by kinematic joints having one or more degrees of freedom. Kinematic joints are points of contact between parts where one of the parts is allowed to move relative to the other. When a joint has a single degree of freedom the part that is allowed to move can only translate in one direction or rotate about one axis. Kinematic variations are the result of the accumulation of dimensional, form, and locational variations of the components.

Prior to manufacturing, an analysis can be performed on the assembly to determine a worst case value or a statistical value for the variation in kinematic variables. Several tolerance accumulation options exist for determining this type of variation, i.e. Worst Case, Root Sum Squares, Six Sigma. All of these are based on first order Taylor's series approximations of assembly functions.

Consider performing a tolerance analysis of a slider-crank mechanism. The mechanism consists of a base, an input disk, and a sliding linkage (figure 2.1).
For this application the rotation of the disk must be synchronized with the position of the slider at several prescribed positions. This is done by rotating the input disk to each prescribed angle and analyzing the assembly in each configuration (figure 2.2). The goal of the tolerance analysis is to estimate the error in the position of the slider due to manufacturing variations in the assembly.

It is important to distinguish between the large scale kinematic motions of a mechanism and the small scale kinematic adjustments due to manufacturing variations. Kinematic motions result from changing a kinematic variable in a mechanism. In the case of the slider-crank mechanism, kinematic motion of the slider occurs when the input disk is rotated through a prescribed angle. The motion is determined by comparing two positions on the slider for the same assembly. On the other hand, kinematic adjustments occur when dimensions deviate from nominal due to manufacturing process variations. The adjustments are determined by comparing each assembly to a perfect assembly. Small kinematic variations occur from one assembly to the next (figure 2.3).

In order to perform a tolerance analysis on a mechanism it must be assembled and analyzed in a series of different configurations. The large motions of the mechanism must
be held at nominal values while analyzing the small kinematic adjustments between mating parts which must accommodate manufacturing variations of individual components.

![Diagram of kinematic motion vs. kinematic adjustments](image)

**Figure 2.3** Kinematic motion vs. kinematic adjustments.

The basic principle will be shown by orienting the assembly in one of the prescribed configurations, where the input disk position is $150^\circ \pm 1^\circ$ from the X axis. In this assembly the independent dimensions with corresponding tolerances are shown in figure 2.4, and the kinematic (dependent) dimensions with unknown variations are shown in figure 2.5.

![Diagram of independent dimensions](image)

**Figure 2.4** Independent dimensions with known tolerances.
2.2 Closed Loops

The basic type of loop used in determining kinematic variations is the closed loop. This consists of a chain of vectors which sum to zero. Closed loops define kinematic constraints on the assembly. The assembly functions mentioned above are developed by forming vector loops through the assembly (figure 2.6).

A vector loop is an engineering model of an assembly. The vectors represent the dependent and independent dimensions. Loops join relevant dimensions into chains. A simple assembly has only one loop, while a complex assembly may have several. The
number of loops is determined by network graph theory, which relates the number of loops to the number of parts and contact points [Larsen 1991].

The vector loop can be written in scalar form, giving equations for the X, Y, and rotation components. By setting them equal to zero, this gives us three equations per loop to solve for the unknown kinematic variations.

\[
\begin{align*}
    h_x &= a \cos(\theta_1) + b \cos(\theta_1 + \theta_2) + c \cos(\theta_1 + \theta_2 + \theta_3) \\
    &\quad + d \cos(\theta_1 + \theta_2 + \theta_3 + \theta_4) = 0 \\
    \quad \text{Equ. 2.1} \\
    h_y &= a \sin(\theta_1) + b \sin(\theta_1 + \theta_2) + c \sin(\theta_1 + \theta_2 + \theta_3) \\
    &\quad + d \sin(\theta_1 + \theta_2 + \theta_3 + \theta_4) = 0 \\
    \quad \text{Equ. 2.2} \\
    h_\theta &= \theta_1 + \theta_2 + \theta_3 + \theta_4 - \theta_5 = 0 \\
    \quad \text{Equ. 2.3}
\end{align*}
\]

In the equations above \( \theta_1 \) is the relative angle which orients vector a with respect to the X axis. \( \theta_2 \) is the relative angle which orients vector b with respect to vector a, and so forth. Defining the assembly model in terms of relative angles is essential to being able to represent the propagation of angular variations. The relative angle describes either a manufactured or a kinematic dimension.

### 2.3 Linearized Equations

As mentioned in section 2.1, the loop equations are expanded using a first order Taylor's series approximation to the equations. The general form of this expansion is:

\[
f = f_0 + \sum \frac{\partial f}{\partial x_i} \, dx_i \\
\quad \text{Equ. 2.4}
\]

Rearranging we can get an equation for the variation in the function.

\[
f - f_0 = \sum \frac{\partial f}{\partial x_i} \, dx_i
\]

\[
df = \sum \frac{\partial f}{\partial x_i} \, dx_i \\
\quad \text{Equ. 2.5}
\]
The general form of scalar loop equations 2.1, 2.2, and 2.3 is:

\[
\begin{align*}
    h_x(x_i,u_j) &= 0 \\
    h_y(x_i,u_j) &= 0 \\
    h_\theta(x_i,u_j) &= 0
\end{align*}
\]  
Equ. 2.6

where:

\(x_i\) = Independent variables, both linear and angular.

\(u_j\) = Dependent variables, both linear and angular.

When equation 2.5 is applied to the general scalar loop equations (Equ.2.6) we get linearized equations for loop variation:

\[
\begin{align*}
    dh_x &= \sum \frac{\partial h_x}{\partial x_i} dx_i + \sum \frac{\partial h_x}{\partial u_j} du_j = 0 \\
    dh_y &= \sum \frac{\partial h_y}{\partial x_i} dx_i + \sum \frac{\partial h_y}{\partial u_j} du_j = 0 \\
    dh_\theta &= \sum \frac{\partial h_\theta}{\partial x_i} dx_i + \sum \frac{\partial h_\theta}{\partial u_j} du_j = 0
\end{align*}
\]  
Equ. 2.7
Equ. 2.8
Equ. 2.9

2.4 Scalar Sensitivities

The partial derivatives of this approximation act as weight factors, or indicate how sensitive the scalar variation is to each variation in the assembly. The partial derivatives will be referred to as scalar sensitivities from here on. There are two types of variables in the linearized equations: translational (a, b, c, d) and rotational (\(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5\)). Translational variables are associated with vector lengths, and rotational variables are associated with vector orientations. These two types of variables lead to corresponding types of scalar sensitivities. The partial derivative can be calculated to determine both translational and rotational scalar sensitivities, but this can get very tedious for rotational scalar sensitivities since angles propagate through the equation. Other simpler methods have been developed that often require only a simple inspection of the drawing [Jinsong 1992].
2.4.1 Translational Scalar Sensitivities

The partial derivatives of equations 2.1 through 2.3 with respect to \( b \) are:

\[
\frac{\partial h_x}{\partial b} = \cos(\theta_1 + \theta_2) \quad \text{Equ. 2.10}
\]

\[
\frac{\partial h_y}{\partial b} = \sin(\theta_1 + \theta_2) \quad \text{Equ. 2.11}
\]

\[
\frac{\partial h_\theta}{\partial b} = 0 \quad \text{Equ. 2.12}
\]

The sum \((\theta_1 + \theta_2)\) is the angle which vector \( b \) makes with the global X axis. When taking partial derivatives with respect to translational variables, the result is just the cosine and sine of the global orientation for the X and the Y loop equations respectively. The X and Y translational scalar sensitivities are just the direction cosines for each vector, which can often be easily retrieved from a CAD drawing. The rotation equation doesn't contain any translational variables so its translational scalar sensitivities are zero.

2.4.2 Rotational Scalar Sensitivities

Rotational variations occur where vectors connect to each other. These connecting points will be called nodes. Let's say that we wanted to find the rotational scalar sensitivity to \( \theta_2 \) in the X equation. This angle is associated with the orientation of all the vectors following \( \theta_2 \) in the vector chain (it appears in the sum of relative angles for each vector). The partial derivative of equation 2.1 with respect to \( \theta_2 \) is:

\[
\frac{\partial h_x}{\partial \theta_2} = -b \sin(\theta_1 + \theta_2) - c \sin(\theta_1 + \theta_2 + \theta_3) - d \sin(\theta_1 + \theta_2 + \theta_3 + \theta_4) \quad \text{Equ. 2.13}
\]

This is just the sum of the negative components in the Y direction from the point of rotation of \( \theta_2 \) to the end of the loop (figure 2.7). (For closed loops the end and the start of the loop are the same point.)
Or, if the coordinates of the end of the loop were (0,0), then $\frac{\partial h_x}{\partial \theta_2}$ would be the Y coordinate of the point of interest (figure 2.8).

\[
\frac{\partial h_y}{\partial \theta_2} = 1
\]

Figure 2.8 Coordinate method for determining horizontal rotational scalar sensitivity.

The partial derivative of equation 2.2 with respect to $\theta_2$ is:

\[
\frac{\partial h_y}{\partial \theta_2} = b \cos(\theta_1+\theta_2) + c \cos(\theta_1+\theta_2+\theta_3) + d \cos(\theta_1+\theta_2+\theta_3+\theta_4) \quad \text{Equ. 2.14}
\]

This is just the sum of the components in the X direction from the point of rotation to the end of the loop (figure 2.9).
Figure 2.9 Graphical interpretation of rotational scalar sensitivity of $h_y$ to $\theta_2$.

Or, if the coordinates of the point of interest were (0,0), then $\partial h_y / \partial \theta_2$ would be the X coordinate of the end of the loop (figure 2.10).

Figure 2.10 The coordinate method for rotational scalar sensitivities using a moving coordinate system.

The problem of having to recalculate coordinates when changing the origin for each new point of interest can be solved by looking at $\partial h_y / \partial \theta_2$ a little differently. Move the origin to the end of the loop as was done for $\partial h_x / \partial \theta_2$ (figure 2.11).
Figure 2.11 The coordinate method for rotational scalar sensitivities using a stationary coordinate system.

The X coordinate now represents the sum of the components in the X direction from the end of the loop to the point of rotation. This is exactly opposite to what \( \frac{\partial h_y}{\partial \theta_2} \) is, but by multiplying by \(-1\) we get the desired results.

The partial derivative of equation 2.3 with respect to \( \theta_2 \) is:

\[
\frac{\partial h_y}{\partial \theta_2} = 1 \quad \text{Equ. 2.15}
\]

because the rotation equation is linear.

In this example:

\[
\frac{\partial h_x}{\partial \theta_2} = \text{the Y coordinate of the point of rotation, relative to the end of the loop.}
\]

\[
\frac{\partial h_y}{\partial \theta_2} = \text{the -X coordinate of the point of rotation, relative to the end of the loop.}
\]

\[
\frac{\partial h_y}{\partial \theta_2} = 1.
\]

But this is a specific case where the rotation of \( \theta_2 \) was positive. If rotation was in the opposite direction and \( \theta_2 \) was negative, the equations would work out to be:

\[
\frac{\partial h_x}{\partial \theta_2} = \text{the -Y coordinate of the point of rotation, relative to the end of the loop.}
\]

\[
\frac{\partial h_y}{\partial \theta_2} = \text{the X coordinate of the point of rotation, relative to the end of the loop.}
\]

\[
\frac{\partial h_y}{\partial \theta_2} = -1.
\]

The rotational scalar sensitivities are dependent on the location of the point of rotation from the end of loop and direction of rotation relative to the previous vector. (The convention used throughout this thesis is counter clockwise being positive.)
When the origin is at the end of the loop:

\[
\frac{\partial h_x}{\partial \theta_i} = Y \text{ coordinate of point of rotation } i \ast \text{ sign of } \theta_i. \quad \text{Eqn. 2.16}
\]

\[
\frac{\partial h_y}{\partial \theta_i} = -X \text{ coordinate of point of rotation } i \ast \text{ sign of } \theta_i. \quad \text{Eqn. 2.17}
\]

\[
\frac{\partial h_\theta}{\partial \theta_i} = 1 \ast \text{ sign of } \theta_i. \quad \text{Eqn. 2.18}
\]

This may seem like a lot of work for this simple example, but if a complex loop is generated using a CAD system, the rotational scalar sensitivities can be found by having the origin located at the end of the loop and retrieving the coordinates of each rotational node and noting the direction of rotation. The main benefit of this method for determining rotational scalar sensitivities is that there is no need to compute partial derivatives numerically or in closed form.

Figure 2.12 shows a more complex example and resulting rotational scalar sensitivities. Once the coordinates and rotation direction of each node in the loop is known, the sensitivities may be determined by inspection.

![Diagram showing rotational scalar sensitivities](image)

Figure 2.12 Using coordinates to determine rotational scalar sensitivities.
Table 2.1 Coordinate Relationship to Rotational Scalar Sensitivities.

<table>
<thead>
<tr>
<th>Node</th>
<th>Node Coordinates (X, Y)</th>
<th>Rotational Scalar Sensitivity in $\theta$ $\frac{\partial h_\theta}{\partial x_i}$</th>
<th>Rotational Scalar Sensitivity in X $\frac{\partial h_x}{\partial x_i}$</th>
<th>Rotational Scalar Sensitivity in Y $\frac{\partial h_y}{\partial x_i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_1$</td>
<td>(0, 0)</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\theta_2$</td>
<td>(0, 2)</td>
<td>-1</td>
<td>-2</td>
<td>0</td>
</tr>
<tr>
<td>$\theta_3$</td>
<td>(1.75, 2)</td>
<td>2</td>
<td>2</td>
<td>-1.75</td>
</tr>
<tr>
<td>$\theta_4$</td>
<td>(2, 2.375)</td>
<td>-1</td>
<td>-2.375</td>
<td>2</td>
</tr>
<tr>
<td>$\theta_5$</td>
<td>(2.375, 2.125)</td>
<td>1</td>
<td>2.125</td>
<td>-2.375</td>
</tr>
<tr>
<td>$\theta_6$</td>
<td>(3.5, 2.125)</td>
<td>-1</td>
<td>-2.125</td>
<td>3.5</td>
</tr>
<tr>
<td>$\theta_7$</td>
<td>(3.5, .75)</td>
<td>-1</td>
<td>-0.75</td>
<td>3.5</td>
</tr>
<tr>
<td>$\theta_8$</td>
<td>(0, .75)</td>
<td>1</td>
<td>0.75</td>
<td>0</td>
</tr>
<tr>
<td>$\theta_9$</td>
<td>(0, 0)</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Many assemblies require more than one vector loop. There will probably not be a common end point for all of the loops, and it would be a lot of work to move the origin to the end of each loop and recalculate the coordinates where rotational scalar sensitivities are required for the analysis. Since we are interested only in the coordinates relative to the end of each loop, whether open or closed, the rotational scalar sensitivities can be calculated using the following equations, no matter how the coordinate system is set up.

\[
\frac{\partial h_x}{\partial \theta_i} = (Y_i - Y_{end}) \times \text{sign of } \theta_i. \quad \text{Equ. 2.19}
\]
\[
\frac{\partial h_y}{\partial \theta_i} = -(X_i - X_{end}) \times \text{sign of } \theta_i. \quad \text{Equ. 2.20}
\]
\[
\frac{\partial h_\theta}{\partial \theta_i} = 1 \times \text{sign of } \theta_i. \quad \text{Equ. 2.21}
\]

### 2.5 Setting Up Equations for Analysis

Using the Direct Linearization Method [Marler 1988] we can combine the sensitivities and variations into a system of equations which can be solved for the dependent variable tolerances. The linearized form of the scalar loop equations was shown to be:

\[
\Sigma \left[ \frac{\partial h_x}{\partial x_i} \right] dx_i + \Sigma \left[ \frac{\partial h_x}{\partial u_j} \right] du_j = 0
\]

\[
\text{Equ. 2.7}
\]

\[
\Sigma \left[ \frac{\partial h_y}{\partial x_i} \right] dx_i + \Sigma \left[ \frac{\partial h_y}{\partial u_j} \right] du_j = 0
\]

\[
\text{Equ. 2.8}
\]

\[
\Sigma \left[ \frac{\partial h_\theta}{\partial x_i} \right] dx_i + \Sigma \left[ \frac{\partial h_\theta}{\partial u_j} \right] du_j = 0
\]

\[
\text{Equ. 2.9}
\]
In matrix form the equations are as follows:

\[ \{dh\} = \begin{bmatrix} \{dx\} \\ \{dU\} \end{bmatrix} + \begin{bmatrix} [A] \\ [B] \end{bmatrix} \{dU\} = \{0\} \]

Equ. 2.25

where:

\[ [A] = \text{Independent scalar sensitivity matrix.} \]

\[ [B] = \text{Dependent scalar sensitivity matrix.} \]

\[ \{dx\} = \text{Part tolerances.} \]

\[ \{dU\} = \text{Kinematic variations.} \]

\[ \{dh\} = \text{Variation of end of loop with respect to the start of loop = \{0\}} \]

To solve for the unknown vector \{dU\}, multiply by \([B]^{-1}\):

\[ \{0\} = [B]^{-1}[A]\{dx\} + [B]^{-1}[B]\{dU\} \]

Equ. 2.26

Rearranging we get:

\[ \{dU\} = -[B]^{-1}[A]\{dx\} \]

Equ. 2.27

Combining \(-[B]^{-1}[A]\) we get:

\[ \{dU\} = [SD]\{dx\} \]

Equ. 2.28

Where \([SD]\) is the sensitivity matrix for the assembly.

The variation in the assembly can be estimated by Worst Case analysis or Root Sum Square (RSS or statistical) analysis as shown in equation 2.29 and 2.30 and compared to the engineering design limits, \(T_{ASM}\), where applicable.

\[ dU_i = \sum |SD_{ij}| dx_j \leq T_{ASM} \]

Equ. 2.29

\[ dU_i = \sqrt{\sum |SD_{ij}| dx_j^2} \leq T_{ASM} \]

Equ. 2.30

2.6 ANSI Y14.5 Form Tolerances for Components

When a designer specifies dimensions and tolerances on a component, it may be necessary to make additional restrictions to assure desired quality. The ANSI Y14.5M-1982 standard makes it possible for designers to specify required form, profile, orientation, and runout feature controls. Since most of the ANSI feature variations are surface or shape variations we will call them form tolerances or controls. Marler [Marler 1988] suggested a zero length vector be added to the vector loop to model a form control. Nominally the
assembly stays the same, but the zero length vector is allowed to vary along with all of the other component and assembly variations. Since form controls are just additional tolerances placed on a component, they are treated as another source of component variation, as far as determining sensitivities goes. There are translational as well as rotational affects depending on the type of form control and how it is applied [Robison 1989].

2.6.1 F Matrix

Features have nominal values of zero, but there still may be some variation. These variations are considered independent since they are controlled at time of manufacture. The form variations accumulate statistically and propagate kinematically just like dimensional variations. We could include their sensitivities and variations in the independent part of the scalar loop equations, but often it is the case that these values are not known with great accuracy. We will separate them so their contribution to the overall assembly variations may be computed separately.

\[
\begin{align*}
\begin{array}{c}
\frac{dh_x}{dx_i} dx_i + \sum \left[\frac{\partial h_x}{\partial \alpha_k} \right] d\alpha_k + \sum \left[\frac{\partial h_x}{\partial u_j} \right] du_j \\
\frac{dh_y}{dx_i} dx_i + \sum \left[\frac{\partial h_y}{\partial \alpha_k} \right] d\alpha_k + \sum \left[\frac{\partial h_y}{\partial u_j} \right] du_j \\
\frac{dh_\theta}{dx_i} dx_i + \sum \left[\frac{\partial h_\theta}{\partial \alpha_k} \right] d\alpha_k + \sum \left[\frac{\partial h_\theta}{\partial u_j} \right] du_j
\end{array}
\end{align*}
\]

Equ. 2.31
Equ. 2.32
Equ. 2.33

Independent Form Dependent

In matrix form the equations are as follows:

\[
\{dh\} = [A]\{dx\} + [F]\{d\alpha\} + [B]\{dU\} = \{0\}
\]

Equ. 2.34

where:

\{dh\} = Variation of end of loop with respect to start of loop = \{0\}

[A] = Independent scalar sensitivity matrix.

\{dx\} = Part tolerances.

[F] = Form scalar sensitivity matrix.

\{d\alpha\} = Form tolerances.

[B] = Dependent scalar sensitivity matrix.

\{dU\} = Kinematic variables tolerances.

To solve for the unknown vector \{dU\}, multiply by \([B]^{-1}\):

\[
\{0\} = [B]^{-1}[A]\{dx\} + [B]^{-1}[F]\{d\alpha\} + [B]^{-1}[B]\{dU\}
\]

Equ. 2.35
Rearranging we get:

\[ \{ dU \} = -[B]^{-1}[A]\{dx\} - [B]^{-1}[F]\{d\alpha\} \quad \text{Equ. 2.36} \]

Combining \(-[B]^{-1}[A]\) and \(-[B]^{-1}[F]\) individually we get:

\[ \{ dU \} = [SD]\{dx\} + [SF]\{d\alpha\} \quad \text{Equ. 2.37} \]

Where \([SD]\) is the dimensional sensitivity matrix for the assembly and \([SF]\) is the form sensitivity matrix for the assembly.

The variation in the assembly can be estimated by Worst Case analysis or a Root Sum Square (RSS or statistical) analysis as shown in equation 2.38 and 2.39 and compared to the engineering design limits, \(T_{ASM}\), where applicable.

\[ dU_i = \sum |SD_{ij}| dx_j + \sum |SF_{ij}| d\alpha_j \leq T_{ASM} \quad \text{Equ. 2.38} \]

\[ dU_i = \sqrt{\sum (SD_{ij} dx_j)^2 + \sum (SF_{ij} d\alpha_j)^2} \leq T_{ASM} \quad \text{Equ. 2.39} \]
Chapter 3 Assembly Specifications

The interaction of parts in an assembly is often important to a product's function. A desired clearance or angle between two parts may need to have a tolerance specified to assure functionality. If a tolerance specification applies to a dimension of a feature on a single component part, it is called a component specification. When more than one part is involved in a tolerance specification, it is called an assembly specification. Commercial CAD systems have built in tools for specifying component tolerances, but there is no standard representation for assembly tolerance specifications. A designer needs a "tool" that will allow him to define assembly tolerance specifications easily, and analyze the assembly before production to see how closely they will be met. This section will introduce a new tolerance specification tool for assemblies which closely parallels existing ANSI Y14.5M-1982 tolerancing methods.

Assembly specifications can be divided into two basic categories: dimensional and geometric form. Each of these categories may have contributions from the accumulation of dimensional and geometric form variations of the individual components, and kinematic variations of the assembly. In order to analyze assembly variations, a model for component tolerance accumulation must be constructed.

3.1 Component vs. Assembly

There are several parallels that can be drawn between individual components and assemblies when looking at variations. The following figures compare different variations that occur in components and in assemblies (figure 3.1 and figure 3.2)
<table>
<thead>
<tr>
<th>Component Tolerances</th>
<th>Assembly Tolerances</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Length &amp; Angle</strong></td>
<td><strong>Length</strong></td>
</tr>
<tr>
<td>$\theta \pm d\theta$</td>
<td>$u \pm du$</td>
</tr>
<tr>
<td>$x \pm dx$</td>
<td></td>
</tr>
</tbody>
</table>

| **Gap**              |                     |
| $u \pm du$           |                     |

| **Angle**            |                     |
| $\theta \pm d\theta$|                     |

Figure 3.1 Dimensional variations in components and assemblies.
<table>
<thead>
<tr>
<th>Component Tolerances</th>
<th>Assembly Tolerances</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Parallelism</strong></td>
<td><strong>Parallelism</strong></td>
</tr>
<tr>
<td>![Diagram of Parallelism]</td>
<td>![Diagram of Parallelism]</td>
</tr>
<tr>
<td><strong>Perpendicularity &amp; Angularity</strong></td>
<td><strong>Perpendicularity &amp; Angularity</strong></td>
</tr>
<tr>
<td>![Diagram of Perpendicularity &amp; Angularity]</td>
<td>![Diagram of Perpendicularity &amp; Angularity]</td>
</tr>
<tr>
<td><strong>Concentricity &amp; Runout</strong></td>
<td><strong>Concentricity &amp; Runout</strong></td>
</tr>
<tr>
<td>![Diagram of Concentricity &amp; Runout]</td>
<td>![Diagram of Concentricity &amp; Runout]</td>
</tr>
<tr>
<td><strong>Position</strong></td>
<td><strong>Position</strong></td>
</tr>
<tr>
<td>![Diagram of Position]</td>
<td>![Diagram of Position]</td>
</tr>
</tbody>
</table>

Figure 3.2 Geometric form variations in components and assemblies.
Figure 3.1 shows the angular and length dimensional variations that can occur in components and assemblies. A "GAP" is just the net length of an assembly. Small adjustments between parts of an assembly lead to kinematic variations. No comparison can be made between components and assemblies for kinematic variations because they can only occur in assemblies. However, they often play a part in the resultant dimensional and geometric form variations of assemblies.

Figure 3.2 shows the variations associated with geometric form controls for components and assemblies. The parallel lines with hatching indicate the tolerance zone associated with each feature. The double box around a geometric form control symbol or datum plane indicates that it refers to an assembly specification, not a geometric form control on a component. An assembly specification will always have a feature on one part referenced to a feature on another part, so only those component geometric form controls which can be referenced to a datum may be applied to assemblies (see Appendix A). Figure 3.3 shows how assembly specifications can be applied to a car door assembly.

Figure 3.3 Example of dimensional and geometric form specifications.
The variation in an assembly needs to be calculated so it can be compared to the assembly specification. For the RSS method of analysis, the comparison is made by assuming the assembly tolerance specified is plus-or-minus 3 sigma and is normally distributed (figure 3.4).

![Diagram of normal distribution with shaded area indicating fraction of out-of-spec assemblies, mean, lower limit (LL), lower tolerance (−3σ), upper limit (UL), and upper tolerance (+3σ).]

Figure 3.4 Limits applied to a normal distribution.

Closed loops are used to find the unknown kinematic variations in an assembly, but an open loop is needed to find variations in an assembly relative to a datum or a point of interest.

3.2 Open Loops

An open loop is a vector loop whose vectors do not sum to zero. Chapter 2 showed how closed loops are used to find resultant kinematic variations. An open loop is used to find the resultant variation in a gap or other assembly feature where the variation measured from one part relative to a datum on another part is to be controlled.

Figure 3.5 shows two assembly specifications added to critical features on a remote positioner device:

1) The position of point P relative to the global X and Y axes.
2) the angular orientation of Part 5 relative to the global X axis.
The open loop describing the variation in position P and angular variation in Part 5 is shown in figure 3.6. Position P is determined by the sum of the vectors. Orientation of Part 5 is determined by the sum of the relative angles of the chain of vectors.
Using the Direct Linearization Method, as was done for the closed loop, the sensitivities and variations can be combined into a system of equations which can be used to solve for the variation in the X direction, the Y direction, and global or relative orientation.

\[
\begin{align*}
\text{dh}_x &= \sum (\frac{\partial h_x}{\partial x_i})dx_i + \sum (\frac{\partial h_x}{\partial u_j})du_j \\
\text{dh}_y &= \sum (\frac{\partial h_y}{\partial x_i})dx_i + \sum (\frac{\partial h_y}{\partial u_j})du_j \\
\text{dh}_\theta &= \sum (\frac{\partial h_\theta}{\partial x_i})dx_i + \sum (\frac{\partial h_\theta}{\partial u_j})du_j \\
\text{Equ. } 3.1 & \quad \text{Equ. } 3.2 & \quad \text{Equ. } 3.3
\end{align*}
\]

In matrix form the equations are as follows:

\[
\{\text{dh}_{OL}\} = [C]\{dx\} + [D]\{dU\} \neq \{0\} \quad \text{Equ. } 3.4
\]

where:

\[
\{\text{dh}_{OL}\} = \text{Variation of end of loop with respect to the start of loop} \neq \{0\} \\
\{dx\} = \text{Part tolerances (independent variables).} \\
\{dU\} = \text{Variations of kinematic variables (dependent variables).} \\
[C] = \text{Independent scalar sensitivity matrix.} \\
[D] = \text{Dependent scalar sensitivity matrix.}
\]

An open loop will not have any dependent variables unless there is at least one closed loop in the assembly. If there is a closed loop in the assembly, then the kinematic variations, \{dU\}, are determined by solving the closed loop equations and then substituting the results into the open loop equation:

\[
\{dU\} = -[B]^{-1}[A]\{dx\} - [B]^{-1}[F]\{d\alpha\} \quad \text{Equ. } 2.36
\]

\[
\{\text{dh}_{OL}\} = [C]\{dx\} + [D]\{dU\} \neq \{0\} \quad \text{Equ. } 3.4
\]

Giving:

\[
\{\text{dh}_{OL}\} = [C]\{dx\} + [D]\{-[B]^{-1}[A]\{dx\} - [B]^{-1}[F]\{d\alpha\}\} \quad \text{Equ. } 3.5
\]

Rearranging gives an open loop equation defining the components of dimensional sensitivity, \{SD\}, and form sensitivity, \{SF\}.

\[
\{\text{dh}_{OL}\} = [C] - [D][B]^{-1}[A]\{dx\} - [D][B]^{-1}[F]\{d\alpha\} \quad \text{Equ. } 3.6
\]

\[
\{\text{dh}_{OL}\} = [SD]\{dx\} + [SF]\{d\alpha\} \quad \text{Equ. } 3.7
\]
If there is not a closed loop in the assembly then the open loop equation simply reduces to:

\[ \{ \text{dh}_{OL} \} = [C]\{ dx \} \]  \hspace{1cm} \text{Equ. 3.8}

which has no dependent variables and \( \{ \text{dh}_{OL} \} \) can be calculated directly.

The \([A]\) matrix of chapter 2 and the \([C]\) matrix contain the independent scalar sensitivities for the closed and open loops respectively. If all the independent variables in the assembly are included in \( dx \), then \([A]\) and \([C]\) will have the same number of columns, but variables that appear in the open loop and not in the closed loops will have a column of zeros in \([A]\), and \textit{vice a versa}.

### 3.3 The \( G \) Matrix

Sometimes there are geometric form controls that are part of the open loop, but not part of the closed loops. The scalar sensitivities for these variations could be included in the \([C]\) matrix since they are independent variables, but will be separated and put in a matrix designated \([G]\). This separation is done for ease of evaluation, just as the separation of the \([A]\) and \([F]\) matrices in chapter 2. Matrix \([G]\) is included in equation 3.4 to give the following general open loop equation:

\[ \{ \text{dh}_{OL} \} = [C]\{ dx \} + [G]\{ d\alpha \} + [D]\{ dU \} \]  \hspace{1cm} \text{Equ. 3.9}

Substituting for the variation of the kinematic variables, \( dU \), gives:

\[ \{ \text{dh}_{OL} \} = [C]\{ dx \} + [G]\{ d\alpha \} + [D]\{ -[B]^{-1}[A]\{ dx \} - [B]^{-1}[F]\{ d\alpha \} \} \]  \hspace{1cm} \text{Equ. 3.10}

And rearranging gives the general open loop equation defining components of dimensional sensitivity, \([SD]\), and form sensitivity, \([SF]\).

\[ \{ \text{dh}_{OL} \} = \left[ [C] - [D][B]^{-1}[A] \right]\{ dx \} + \left[ [G] - [D][B]^{-1}[F] \right]\{ d\alpha \} \]  \hspace{1cm} \text{Equ. 3.11}

\[ \{ \text{dh}_{OL} \} = [SD]\{ dx \} + [SF]\{ d\alpha \} \]  \hspace{1cm} \text{Equ. 3.12}

Next, several specific assembly specifications are presented to show how they are modeled.
3.4 Specification of Assembly Dimensions

3.4.1 Angular Assembly specifications

An angular specification allows a tolerance to be specified for the nominal angle of a plane or an axis with respect to a reference plane or axis. For example, the angle $\Phi$ in figure 3.7 is specified relative to a horizontal datum plane. The magnitude of $\Phi$, however, is the result of manufactured angles $\theta_1$, $\theta_2$, and $\theta_3$. The variation of $\Phi$ is the result of variations in $\theta_1$, $\theta_2$, and $\theta_3$.

![Diagram of angular assembly specification](image)

Figure 3.7 Angular assembly specification.

An open loop is created from the reference plane or axis to the angled plane or axis (figure 3.8) and the variation in the relative orientation $d\Phi$ is calculated using the method described for open loop analysis.
This variation can also be looked at as an accumulation of all the angular component and kinematic variations from the start to the end of the loop.

Worst Case:

\[ dh_0 = \sum |d\theta_i| \quad \text{Eqn. 3.13} \]

\[ d\Phi = d\theta_1 + d\theta_2 + d\theta_3 \]

RSS:

\[ dh_0 = \sqrt{\sum (d\theta_i)^2} \quad \text{Eqn. 3.14} \]

\[ d\Phi = \sqrt{(d\theta_1)^2 + (d\theta_2)^2 + (d\theta_3)^2} \]

The nominal angle is calculated by summing the relative angles between adjacent vectors, starting from the +X axis (counterclockwise is a positive rotation).

\[ \theta_{\text{nominal}} = \sum d\theta_i \quad \text{Eqn. 3.15} \]

\[ \Phi = \theta_1 + \theta_2 - 180^\circ + \theta_3 \]

The calculated nominal angle and variation are then compared to the specified nominal angle and variation to determine the probability that the assembly will meet the specification. If the nominal values are not equal, then the mean of the distribution has shifted. If the
calculated variation \( d\Phi \) is greater than the specified design limits, the tolerances \( d\theta_1 \), \( d\theta_2 \), or \( d\theta_3 \) will need to be reduced.

3.4.2 Length Assembly Specifications

A length specification allows a tolerance to be specified for the nominal distance between points on two parts.

![Figure 3.9 Length assembly specification.](image)

An open loop is created from the point on the reference part to the point on the part where the distance is to be controlled (figure 3.10).

![Figure 3.10 Open loop for length assembly specification.](image)

In this example there are no closed loops so \([SD]\) equals \([C]\), where \([C]\) is made up of all the pertinent sensitivities in the assembly.
These values are used in forming the expressions for the Worst Case and the RSS variations. Since we are only interested in \( \Delta h_x \), we only need the first row of \([SD]\).

\[
\begin{align*}
\text{W.C.: } \Delta L &= \Delta h_x = |10| \cdot da + |a| \cdot d\theta_1 + |1| \cdot db + |l-a| \cdot d\theta_2 + |l\cos(\theta_2)| \cdot dc \\
\text{RSS: } \Delta L &= \Delta h_x = \sqrt{(0 \cdot da)^2 + (a \cdot d\theta_1)^2 + (1 \cdot db)^2 + (-a \cdot d\theta_2)^2 + (\cos(\theta_2) \cdot dc)^2}
\end{align*}
\]

The variation in the \( X \) direction is then compared to the specification. If the dimensional length specification is along the \( Y \) axis then the variation in the \( Y \) direction is used for comparison.

If the dimensional length specification does not lie along the \( X \) or the \( Y \) axes then the variations in the \( X \) and the \( Y \) directions must be transformed to correspond to a variation in the direction of the length. There are two methods that can be followed for this type of analysis: 1) Geometric Transformation and 2) Sensitivity Transformation.

The Geometric Transformation method rotates the entire assembly so the length direction becomes the new \( X \) axis (figure 3.11).

![Figure 3.11 Geometric Transformation method for determining off-axis length variation.](image)
This requires a recalculation of the scalar sensitivities, which are then used to calculate the variation in the X direction.

The Sensitivity Transformation method applies a rotational transformation to the existing dimensional and form sensitivity matrices, which are used to calculate the variation in the X direction.

\[
\{dh\}_{\text{length}} = [R][SD]\{dx\} + [R][SF]\{d\alpha\}
\]

Equ. 3.16

\[
[R] = \begin{bmatrix}
\cos\theta & -\sin\theta & 0 \\
\sin\theta & \cos\theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Where \(\theta\) is the angle the length makes with the X axis. Since we are only interested in the variation of the length, we can simplify the calculations by solving only for \(dh_X\) and only using the first row of \([R]\).

\[
dh_X = [\cos\theta \ -\sin\theta \ 0][SD]\{dx\} + [\cos\theta \ -\sin\theta \ 0][SF]\{d\alpha\}
\]

Equ. 3.17

Caution should be taken not to apply the rotational transformation matrix after \(dh\)_{length} is calculated using the original axis orientation because this will produce incorrect results for the variation of the length.

### 3.5 Geometric Form Assembly Specifications

#### 3.5.1 Angularity, Perpendicularity, and Parallelism

An angularity specification consists of a plane or axis that has a nominal orientation relative to a datum plane or axis, and a tolerance zone of parallel planes offset on either side of the plane or axis. The total zone width is the specified tolerance (figure 3.12).
Figure 3.12 Angularity specification.

An open loop is created from the datum to the angled plane or axis and the variation in the relative orientation $dh_0$ is calculated using the method described for open loop analysis. This variation can also be looked at as an accumulation of all the angular component and kinematic variations from the start to the end of the loop (figure 3.13).

Figure 3.13 Angles and variations that contribute to the specified angularity.

Worst Case:

$$dh_0 = \sum l d\theta_i$$
$$d\Phi = d\theta_1 + d\theta_2$$

Equation 3.18

RSS:

$$dh_0 = \sqrt{\sum (d\theta_i)^2}$$
$$d\Phi = \sqrt{(d\theta_1)^2 + (d\theta_2)^2}$$

Equation 3.19

The nominal angle is also calculated.

$$\theta_{\text{nominal}} = \sum d\theta_i$$
$$\Phi = 180^\circ - \theta_1 - \theta_2$$

Equation 3.20
Figure 3.14 Tolerance zone and characteristic length for angularity specification.

A characteristic length of the final plane or axis is used along with the width of the tolerance zone (figure 3.14) to determine the allowable plus or minus angular variation (figure 3.15).

Figure 3.15 Angular variation allowed with specified tolerance zone width and characteristic length of link.
Since the tolerance zone width is so small compared to the characteristic length the allowable angular variation is approximated as follows:

$$\text{Angular variation allowed} \approx \frac{\text{zone width}}{\text{characteristic length}} = \frac{.01}{12.5} = 0.0008 \text{ radians}$$

The specified nominal angle and variation are then compared to the calculated nominal angle and variation to determine the probability that the assembly will meet the specification. If the nominal values are not equal then there is a shift in the distribution. Perpendicularity and parallelism are just special cases of angularity, where the nominal specified angle is 90° and 0° respectively.

3.5.2 True Position

True position allows a designer to specify the limits on the variation in position of one point relative to a reference point usually consisting of two datums (figure 3.16).

![Figure 3.16 True position.](image)

There is a circular tolerance zone whose diameter is specified (figure 3.17).
An open loop starts at the datums and ends at the point where position is to be controlled. The translational variations as well as the angular component and kinematic variations contribute to the assembly X and Y translational variations (figure 3.18) and are calculated as explained earlier.

These variations are resolved into a diameter (figure 3.19) which is then compared to the specified tolerance zone.
3.5.3 Concentricity and Runout

Concentricity and runout can be analyzed in the same manner as perpendicularity, length, or true position, depending on whether the axis is in the 2-D plane or normal to the plane. If the axis is in the plane (Figure 3.20) then the concentricity analysis is performed in the same manner as length or gap, and the runout analysis is performed in the same manner as perpendicularity.

![Diagram showing concentricity and runout](image)

**Figure 3.20 Axis in 2-D plane.**

If the axis is normal to the plane (Figure 3.21) then the variation associated with a runout specification is the same as the variation associated with a concentricity specification, and the analysis is performed in the same manner as true position for both.
3.6 Closed Loop Option for Kinematic Variable Specifications

In the preceding sections it was shown how open loops were used to calculate assembly variations. This will always work, but the addition of an open loop may be unnecessary if the specification is on a kinematic variable. Kinematic variations are solved in closed loops, so the comparison can be made directly to the specification without building an open loop and solving for the variation.
Chapter 4  Example Problem:  One Way Clutch

4.1 The Problem

A one way clutch similar to the kind found in pull-start gas lawn mowers is to be designed and is made from four basic components: a hub, rollers, springs, and a ring. When assembled, the rollers remain in contact with the hub and the ring by the force of the springs (figure 4.1).

![Diagram of one way clutch assembly](image)

Figure 4.1 One way clutch assembly.

This assembly is designed to allow rotation in only one direction. The hub is attached to a shaft, which in turn is attached to a driving mechanism. When the hub rotates clockwise relative to the ring, the rollers slip on the inside of the ring. If the hub rotates counterclockwise then the spring allows the rollers to wedge between the hub and the ring, causing the two to lock and rotate together. The clutch must also be able to release when the hub is rotated clockwise again. To assure that the assembly will lock and release properly the wedge angle, $\Phi_1$ (figure 4.2), must be less than $9^\circ$ and greater than $5^\circ$. Since $\Phi_1$ is an assembly variable we will assign it an assembly specification of $7^\circ \pm 2^\circ$. 
Figure 4.2 An assembly specification of $7^\circ \pm 2^\circ$ is placed on the wedge angle.

Given the tolerances for the components, an assembly variation for $\Phi_1$ needs to be calculated and compared to the assembly specification to determine functionality.

4.2 The Model

The one-way clutch problem illustrates how a closed vector loop may be used to find the resultant tolerance of a crucial assembly variable. Also, due to symmetry, only one vector loop is necessary to model the assembly in order to calculate all of the kinematic variables. The following figure shows the vector loop and variables that will be used (figure 4.3).
The loop equations for this model are:

\[
\begin{align*}
h_x &= b \cos(0°) + c \cos(90°) + c \cos(90° - \Phi_1) \\
&\quad + e \cos(90° - \Phi_1 + 180°) + a \cos(90° - \Phi_1 + 180° - \Phi_2) = 0 \quad \text{Equ. 4.1} \\
h_y &= b \sin(0°) + c \sin(90°) + c \sin(90° - \Phi_1) \\
&\quad + e \sin(90° - \Phi_1 + 180°) + a \sin(90° - \Phi_1 + 180° - \Phi_2) = 0 \quad \text{Equ. 4.2} \\
\theta &= 0° + 90° - \Phi_1 + 180° - \Phi_2 - 90° = 0 \quad \text{Equ. 4.3}
\end{align*}
\]

Some of the angles in the equations just show the nominal value and will not have variations associated with them. This is because the type of contact naturally produces angles of 90° or 180°, or the variations will be taken care of by applying form controls (figure 4.4).
The form controls have nominal dimensions of zero, but still have variations associated with them. The variations for flatness and circularity in this problem are applied normal to the surface of contact due to the type of contact between parts.

Scalar sensitivities are determined using direction cosines, coordinates, and relative rotation directions. The direction cosine method was used for the variables that have translational variations, and the coordinate method was used for the variables that have rotational variations. Figure 4.5 shows the coordinates of the two points at which rotational variations occur. Table 4.1 shows the resulting translational sensitivities. Table 4.2 shows the resulting rotational sensitivities.
Table 4.1 Translational Scalar Sensitivities.

<table>
<thead>
<tr>
<th>Vector</th>
<th>Nominal Global Angle ($\theta_i$)</th>
<th>$\cos(\theta_i)$</th>
<th>$\sin(\theta_i)$</th>
<th>Independent / Form or Dependent</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>0°</td>
<td>1</td>
<td>0</td>
<td>Independent / Form or Dependent</td>
</tr>
<tr>
<td>$\alpha_1$</td>
<td>90°</td>
<td>0</td>
<td>1</td>
<td>Independent / Form</td>
</tr>
<tr>
<td>$\alpha_2$</td>
<td>90°</td>
<td>0</td>
<td>1</td>
<td>Independent / Form</td>
</tr>
<tr>
<td>C</td>
<td>90°</td>
<td>0</td>
<td>1</td>
<td>Independent</td>
</tr>
<tr>
<td>$\alpha_3$</td>
<td>83°</td>
<td>0.12187</td>
<td>0.99255</td>
<td>Independent / Form</td>
</tr>
<tr>
<td>$\alpha_4$</td>
<td>83°</td>
<td>0.12187</td>
<td>0.99255</td>
<td>Independent / Form</td>
</tr>
<tr>
<td>E</td>
<td>-97°</td>
<td>-0.12187</td>
<td>-0.99255</td>
<td>Independent</td>
</tr>
<tr>
<td>A</td>
<td>90°</td>
<td>0</td>
<td>1</td>
<td>Independent</td>
</tr>
</tbody>
</table>

Figure 4.5 Coordinates of rotational variations.

Table 4.2 Rotational Scalar Sensitivities.

<table>
<thead>
<tr>
<th>Angle</th>
<th>Rotation Sign</th>
<th>$\text{Sign} \times Y_{\text{coord}}$</th>
<th>$\text{Sign} \times (-X_{\text{coord}})$</th>
<th>Independent / Form or Dependent</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Phi_1$</td>
<td>-1</td>
<td>-11.43</td>
<td>4.798</td>
<td>Dependent</td>
</tr>
<tr>
<td>$\Phi_2$</td>
<td>-1</td>
<td>27.645</td>
<td>0</td>
<td>Dependent</td>
</tr>
</tbody>
</table>
4.3 Sample Analysis Procedure

Using the Direct Linearization Method we can combine the sensitivities and variations into a system of equations which can be solved for the dependent variable tolerances, \( \{dU\} \).

\[
\{dU\} = [SD]\{dx\} + [SF]\{d\alpha\} \quad \text{Eqn. 2.34}
\]

where \([SD]\) is the dimensional sensitivity matrix for the assembly calculated from \(-[B]^{-1}[A]\), and \([SF]\) is the form sensitivity matrix for the assembly calculated from \(-[B]^{-1}[F]\).

\[
\{dU\} = -[B]^{-1}[A]\{dx\} - [B]^{-1}[F]\{d\alpha\} \quad \text{Eqn. 2.33}
\]

The matrices are made up of values found in tables 4.1 and 4.2.

Independent scalar sensitivity matrix:

\[
[A] = \begin{bmatrix}
0.12187 & -0.12187 & 0 \\
1.99255 & -0.99255 & 1.0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Dependent scalar sensitivity matrix:

\[
[B] = \begin{bmatrix}
1.0 & -11.43 & 27.645 \\
0 & 4.798 & 0 \\
0 & -1.0 & -1.0
\end{bmatrix}
\]

Form scalar sensitivity matrix:

\[
[F] = \begin{bmatrix}
0 & 0 & 0.12187 & 0.12187 \\
1.0 & 1.0 & 0.99255 & 0.99255 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Part tolerances:

\[
\{dx\} = \begin{bmatrix}
0.01 \\
0.0125 \\
0.05
\end{bmatrix}
\]

The form tolerances are converted to a plus or minus symmetrical tolerance by dividing the specified tolerance zone by 2.
Form tolerances:

\[
\{d\alpha\} = \begin{bmatrix}
0.0125 \\
0.00015 \\
0.0015 \\
0.005
\end{bmatrix}
\]

The dimensional sensitivity matrix, \([SD]\), and the form sensitivity matrix, \([SF]\), are calculated using standard linear algebra methods:

\[
\begin{bmatrix}
C \\
E \\
A
\end{bmatrix} = \begin{bmatrix}
B & -16.3492 & 8.2052 & -8.1440 \\
-0.4153 & 0.2069 & 0.2084 \\
0.4153 & -0.2069 & 0.2084
\end{bmatrix}
\]

\[
\Phi_1 = \begin{bmatrix}
-8.1440 & -8.1440 & -8.2052 & -8.2052 \\
-0.2084 & -0.2084 & -0.2069 & -0.2069 \\
0.2084 & 0.2084 & 0.2069 & 0.2069
\end{bmatrix}
\]

Equation 2.34 may be used to form a tolerance accumulation expression for Worst Case or a Root Sum Square (RSS or statistical) tolerance analysis as shown in equations 2.35 and 2.36 and compared to the engineering design limits, \(T_{ASM}\), where applicable.

\[
dU_i = \Sigma|SD_{ij}| \cdot dx_j + \Sigma|SF_{ij}| \cdot d\alpha_j \leq T_{ASM}
\text{ Equ. 2.35}
\]

\[
dU_i = \sqrt{\Sigma[SD_{ij} \cdot dx_j]^2 + \Sigma[SF_{ij} \cdot d\alpha_j]^2} \leq T_{ASM}
\text{ Equ. 2.36}
\]

Substituting the values from above yields estimates for the assembly variations:

\[
\{dU\} = \begin{bmatrix}
d_b \\
d\Phi_1 \\
d\Phi_2
\end{bmatrix} = \begin{bmatrix}
0.4641 \\
0.018567 \text{ rad} \\
0.018567 \text{ rad}
\end{bmatrix} = \begin{bmatrix}
0.4641 \\
0.6793^* \\
0.6793^*
\end{bmatrix} = \begin{bmatrix}
0.021422 \text{ rad} \\
0.021422 \text{ rad}
\end{bmatrix} = \begin{bmatrix}
0.8406 \\
1.2274^*
\end{bmatrix}
\]

\[
\text{RSS} \quad \text{WC}
\]

The design specification was set to control \(\Phi_1\), that is \(T_{ASM} \leq 2^*\). The predicted assembly variation, \(d\Phi_1 = 0.6793^*\), is considerably less than \(2^*\) so the assembly design specification has a very high probability of being met. Statistically, there will be less than one reject per million assemblies manufactured, so tolerances could be relaxed to cut down on manufacturing costs.
Chapter 5 Example Problem: Remote Positioner

5.1 The Problem

A remote positioning mechanism used in wind tunnel testing is to be designed from a set of linkages connected by pins and attached to the wind tunnel (figure 5.1).

![Remote Positioner Assembly](image)

Figure 5.1 Remote positioner assembly.

This assembly is designed to allow the orientation of the arm in the tunnel to be adjusted linearly by the input linkage located outside the tunnel. The angle of the arm will be oriented to several different angles during wind tunnel testing, so a tolerance analysis should be performed with the arm in several positions. This example will only look at the case where the input linkage is rotated 180° from the positive X axis. The allowable relative variation between the input linkage and the arm in the tunnel will be controlled by a parallelism assembly feature control specification of 0.2. The position of the end of the arm in the tunnel is also important to the tests which will be performed, and will be controlled by a true position assembly feature control specification of 1.0. The input angle ($\theta_1$, figure 5.3) has a variation also, so the global orientation of the arm will be controlled by an angularity assembly feature control specification of 0.3 (figure 5.2).
Figure 5.2 Parallelism and true position assembly feature controls.

Given the tolerances for the components, an assembly variation for the relative angle and position of the end of the arm will be calculated and compared to the assembly specifications to determine functionality.

5.2 The Model

The remote positioner problem illustrates how closed loops are used to calculate the kinematic variations and how open loops are used to estimate the relative variations of critical assembly parameters. The following figure shows the vector loops and variables that will be analyzed (figure 5.3).
Figure 5.3 Vector loops for the remote positioner.

Closed loops 1 and 2 were used to describe the kinematic closure constraints on the assembly. The nominal values of the six kinematic variables, $\Phi_1$ through $\Phi_6$, are known, but their variations due to manufacturing tolerances must be determined analytically. Open loop 1 was used to analyze the variation in the position of point P at the end of the arm.
relative to the global X and Y axes. Open loop 2 was used to analyze the angular variation of the arm relative to the input link.

The following tables show the scalar sensitivities used to determine the assembly variations. The direction cosine method is used for the variables that have translational variations, and the coordinate method is used for the variables that have rotational variations.

Table 5.1 Translational Scalar Sensitivities.

<table>
<thead>
<tr>
<th>Vector</th>
<th>Nominal Global Angle ($\theta_i$)</th>
<th>Cos($\theta_i$)</th>
<th>Sin($\theta_i$)</th>
<th>Independent / Form or Dependent</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>$180^\circ$</td>
<td>-1</td>
<td>0</td>
<td>Independent</td>
</tr>
<tr>
<td>b</td>
<td>$-120.25^\circ$</td>
<td>-0.50377</td>
<td>-0.86384</td>
<td>Independent</td>
</tr>
<tr>
<td>c</td>
<td>$0^\circ$</td>
<td>1</td>
<td>0</td>
<td>Independent</td>
</tr>
<tr>
<td>d</td>
<td>$59.75^\circ$</td>
<td>0.50377</td>
<td>0.86384</td>
<td>Independent</td>
</tr>
<tr>
<td>e</td>
<td>$-90^\circ$</td>
<td>0</td>
<td>-1</td>
<td>Independent</td>
</tr>
<tr>
<td>f</td>
<td>$-137.354^\circ$</td>
<td>-0.73555</td>
<td>-0.67747</td>
<td>Independent</td>
</tr>
<tr>
<td>g</td>
<td>$90^\circ$</td>
<td>0</td>
<td>1</td>
<td>Independent</td>
</tr>
<tr>
<td>i</td>
<td>$42.646^\circ$</td>
<td>0.73555</td>
<td>0.67747</td>
<td>Independent</td>
</tr>
<tr>
<td>j</td>
<td>$0^\circ$</td>
<td>1</td>
<td>0</td>
<td>Independent</td>
</tr>
</tbody>
</table>

For the rotational scalar sensitivities the coordinates of the points at which the rotational variations occur were used. Since there was more than one loop in the model and all their end points didn't coincide with the global origin, the most general form for determining rotational scalar sensitivities was used. The calculations used (0,0) and (-27.2392,-8.9839) as the global end point coordinates for closed loops 1 and 2 respectively. The global end point coordinates (-5.2392,-58.2839) were used for both open loops because both end points were located at the same global position.

Table 5.2 Rotational Scalar Sensitivities for Closed Loop 1.

<table>
<thead>
<tr>
<th>Angle</th>
<th>Rotation Sign</th>
<th>Sign*(Y_{coord-Y_{end}})</th>
<th>Sign*(-X_{coord-X_{end}})</th>
<th>Independent / Form or Dependent</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_1$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>Independent</td>
</tr>
<tr>
<td>$\theta_2$</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>Independent</td>
</tr>
<tr>
<td>$\phi_1$</td>
<td>1</td>
<td>0</td>
<td>22</td>
<td>Dependent</td>
</tr>
<tr>
<td>$\phi_2$</td>
<td>1</td>
<td>-8.9839</td>
<td>27.2392</td>
<td>Dependent</td>
</tr>
<tr>
<td>$\phi_3$</td>
<td>1</td>
<td>-8.9839</td>
<td>5.2392</td>
<td>Dependent</td>
</tr>
</tbody>
</table>
Table 5.3 Rotational Scalar Sensitivities for Closed Loop 2.

<table>
<thead>
<tr>
<th>Angle</th>
<th>Rotation Sign</th>
<th>( \text{Sign}^*(Y_{\text{coord}} - Y_{\text{end}}) )</th>
<th>( \text{Sign}^*(-X_{\text{coord}} - X_{\text{end}}) )</th>
<th>Independent / Form or Dependent</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta_3 )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>Independent</td>
</tr>
<tr>
<td>( \theta_4 )</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>Independent</td>
</tr>
<tr>
<td>( \phi_2 )</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>Dependent</td>
</tr>
<tr>
<td>( \phi_4 )</td>
<td>-1</td>
<td>49.3</td>
<td>0</td>
<td>Dependent</td>
</tr>
<tr>
<td>( \phi_5 )</td>
<td>-1</td>
<td>58.0393</td>
<td>-9.4886</td>
<td>Dependent</td>
</tr>
<tr>
<td>( \phi_6 )</td>
<td>-1</td>
<td>8.7393</td>
<td>-9.4886</td>
<td>Dependent</td>
</tr>
</tbody>
</table>

Table 5.4 Rotational Scalar Sensitivities for Open Loop 1.

<table>
<thead>
<tr>
<th>Angle</th>
<th>Rotation Sign</th>
<th>( \text{Sign}^*(Y_{\text{coord}} - Y_{\text{end}}) )</th>
<th>( \text{Sign}^*(-X_{\text{coord}} - X_{\text{end}}) )</th>
<th>Independent / Form or Dependent</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta_1 )</td>
<td>1</td>
<td>58.2839</td>
<td>-5.2392</td>
<td>Independent</td>
</tr>
<tr>
<td>( \theta_3 )</td>
<td>1</td>
<td>49.3</td>
<td>22</td>
<td>Independent</td>
</tr>
<tr>
<td>( \theta_5 )</td>
<td>1</td>
<td>0</td>
<td>22</td>
<td>Independent</td>
</tr>
<tr>
<td>( \phi_1 )</td>
<td>1</td>
<td>58.2839</td>
<td>16.7608</td>
<td>Dependent</td>
</tr>
<tr>
<td>( \phi_4 )</td>
<td>-1</td>
<td>0</td>
<td>-22</td>
<td>Dependent</td>
</tr>
</tbody>
</table>

Table 5.5 Rotational Scalar Sensitivities for Open Loop 2.

<table>
<thead>
<tr>
<th>Angle</th>
<th>Rotation Sign</th>
<th>( \text{Sign}^*(Y_{\text{coord}} - Y_{\text{end}}) )</th>
<th>( \text{Sign}^*(-X_{\text{coord}} - X_{\text{end}}) )</th>
<th>Independent / Form or Dependent</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta_3 )</td>
<td>1</td>
<td>49.3</td>
<td>22</td>
<td>Independent</td>
</tr>
<tr>
<td>( \theta_5 )</td>
<td>1</td>
<td>0</td>
<td>22</td>
<td>Independent</td>
</tr>
<tr>
<td>( \phi_1 )</td>
<td>1</td>
<td>58.2839</td>
<td>16.7608</td>
<td>Dependent</td>
</tr>
<tr>
<td>( \phi_4 )</td>
<td>-1</td>
<td>0</td>
<td>-22</td>
<td>Dependent</td>
</tr>
</tbody>
</table>

5.3 Sample Analysis Procedure

The sensitivities and variations are combined into a system of linearized equations which can be solved for the variations in an open loop.

\[
\{dh_{OL}\} = [SD]\{dx\} + [SF]\{d\alpha\} \quad \text{Equ. 3.7}
\]

where [SD] is the dimensional sensitivity matrix for the assembly calculated from
[C]−[D][B]−1[A]. There are no form tolerances in this example so equation 3.7 can be written as:

$$(dh_{OL}) = [[C]−[D][B]−1[A]][dx]$$

Equation 5.1

Care must be taken when constructing the [B] matrix because the rotation of $\phi_2$ is positive in Closed Loop 1 and negative in Closed Loop 2 (see Appendix B for all matrices). The dimensional sensitivity matrix for Open Loop 1 is shown below.

$$[SD_1](dx) =$$

$$\begin{bmatrix}
0.0087266 & 0.005 & 0.005 & 0.005 & 0.005 & 0.0087266 \\
0.005 & 0.005 & 0.005 & 0.005 & 0.005 & 0.0087266 \\
0.005 & 0.005 & 0.005 & 0.005 & 0.005 & 0.0087266 \\
0.005 & 0.005 & 0.005 & 0.005 & 0.005 & 0.0087266 \\
0.005 & 0.005 & 0.005 & 0.005 & 0.005 & 0.0087266 \\
0.005 & 0.005 & 0.005 & 0.005 & 0.005 & 0.0087266 \\
\end{bmatrix}$$

Equation 5.1 may be used to form a tolerance accumulation expression for Worst Case or Root Sum Square (RSS or statistical) tolerance analysis as shown in equation 5.2 and 5.3 and compared to the engineering design limits, $T_{ASM}$.

$$dh_{OLi} = \sum |SD_{ij}| dx_j \leq T_{ASM}$$

Equation 5.2
\[ dx_{OL1} = 10i \cdot 0.0087266 \text{ rad} + 15.4876i \cdot 0.005 + 12.7645i \cdot 0.005 + i \cdot 6.4876i \cdot 0.005 + \]
\[ i \cdot 3.2683i \cdot 0.005 + 58.2839r \cdot 0.0087266 \text{ rad} + 10i \cdot 0.005 + 149.31r \cdot 0.0087266 \text{ rad} + \]
\[ 10i \cdot 0.005 + 10i \cdot 0.005 + 10i \cdot 0.005 + 10i \cdot 0.0087266 \text{ rad} + 10i \cdot 0.0087266 \text{ rad} + \]
\[ 11i \cdot 0.005 = 1.0339 \]

\[ dh_{OLi} = \sqrt{\sum SD_{ij} dx_j^2} \leq T_{ASM} \quad \text{Equ. 5.3} \]

\[ dx_{OL1} = (0 \cdot 0.0087266 \text{ rad})^2 + (5.4876 \cdot 0.005)^2 + (2.7645 \cdot 0.005)^2 + (-6.4876 \cdot 0.005)^2 + \]
\[ (-3.2683 \cdot 0.005)^2 + (58.2839 \cdot 0.0087266 \text{ rad})^2 + (0 \cdot 0.005)^2 + (49.31 \cdot 0.0087266 \text{ rad})^2 + \]
\[ (0 \cdot 0.005)^2 + (0 \cdot 0.005)^2 + (0 \cdot 0.005)^2 + (0 \cdot 0.0087266 \text{ rad})^2 + (0 \cdot 0.0087266 \text{ rad})^2 + \]
\[ (1 \cdot 0.005)^2)^.5 = 0.6679 \]

The plus-or-minus variations in open loops 1 and 2, \{dh_{OL}\}, were estimated using the WC method:

\[
\{dh_{OL1}\} = \begin{bmatrix} dx_{OL1} \\ dy_{OL1} \\ d\theta_{OL1} \end{bmatrix} = \begin{bmatrix} 1.0339 \\ 0.4779 \\ 0.0287 \text{ rad} \end{bmatrix} = \begin{bmatrix} 1.0339 \\ 0.4779 \\ 1.6466^* \end{bmatrix} 
\]

\[
\{dh_{OL2}\} = \begin{bmatrix} dx_{OL2} \\ dy_{OL2} \\ d\theta_{OL2} \end{bmatrix} = \begin{bmatrix} 1.5475 \\ 0.5236 \\ 0.02001 \text{ rad} \end{bmatrix} = \begin{bmatrix} 1.5475 \\ 0.5236 \\ 1.1466^* \end{bmatrix} 
\]

and the RSS method:

\[
\{dh_{OL1}\} = \begin{bmatrix} dx_{OL1} \\ dy_{OL1} \\ d\theta_{OL1} \end{bmatrix} = \begin{bmatrix} 0.6679 \\ 0.2763 \\ 0.01515 \text{ rad} \end{bmatrix} = \begin{bmatrix} 0.6679 \\ 0.2763 \\ 0.8679^* \end{bmatrix} 
\]

\[
\{dh_{OL2}\} = \begin{bmatrix} dx_{OL2} \\ dy_{OL2} \\ d\theta_{OL2} \end{bmatrix} = \begin{bmatrix} 0.8397 \\ 0.2801 \\ 0.01238 \text{ rad} \end{bmatrix} = \begin{bmatrix} 0.8397 \\ 0.2801 \\ 0.7094^* \end{bmatrix} 
\]

The variations in Open Loop 1 need to be converted to variations that can be compared to the angularity and true position specifications. The angular variation of Open Loop 1 is converted to a width for comparison to the tolerance zone by using the characteristic length of the arm:

Width = characteristic length \cdot \sin(2 \cdot d\theta_{OL1})

Width_{WC} = 1.2621 \quad \text{Width}_{RSS} = 0.6664
Both of these methods predict that the angular variation will exceed the 0.3 angularity specification. In order to determine the approximate number of rejects we must see what is happening with the distributions. Before we converted the angular variation to a width variation, we assumed that we had a normal distribution. By using a non-linear conversion equation we ended up with something that was no longer normally distributed, and the plus-or-minus angular variation was converted to a one-sided distribution. The distribution is one-sided because you cannot have a negative tolerance zone for the width. A positive angular variation gives a positive width, and a negative angular variation gives a positive width also. To avoid this problem, the number of rejects is approximated by converting the limits on the width to angular limits and comparing them to the angular variation. Even this approximation is guess work because no one has characterized the distribution for angular variation.

\[
\text{Angle}_{\text{spec}} = \sin^{-1}\left(\frac{\text{Width}_{\text{spec}}}{\text{Characteristic length}}\right) = \sin^{-1}\left(\frac{0.3}{22}\right) = 0.7813^\circ
\]

This gives an upper limit of \(0.7813^\circ\) and a lower limit of \(-0.7813^\circ\) as shown in figure 5.4.

![Figure 5.4 Comparison of assembly variation to specification.](image)

The RSS method predicts approximately 6,917 rejects per 1 million assemblies. In this case, several component tolerances would have to be tightened to meet the specification. The Worst Case method would require extreme reductions in those components with the largest sensitivities. However, Worst Case is overly conservative. The RSS analysis would allow more moderate decreases.
The X and Y variations of loop 1 are converted to a diameter for comparison to the circular tolerance zone of true position:

\[
\text{Diameter} = 2 \sqrt{dx_{OL}^2 + dy_{OL}^2}
\]

\[
\text{Diameter}_{WC} = 1.1390 \quad \text{Diameter}_{RSS} = 0.7228
\]

While the worst case method predicts the position variation will exceed the 1.0 true position specification, the RSS method predicts it will be less than the limit with only 33 rejects per 1 million assemblies.

The angular variation of open loop 2 is converted to a width for comparison to the tolerance zone by using the characteristic length of the arm as was done for open loop 1:

\[
\text{Width} = \text{characteristic length} \cdot \sin(2 \cdot d\theta_{OL2})
\]

\[
\text{Width}_{WC} = 0.8803 \quad \text{Width}_{RSS} = 0.5447
\]

Both of these methods predict that the angular variation will exceed the 0.2 parallelism specification.

\[
\text{Angle}_{spec} = \sin^{-1}\left(\frac{\text{Width}_{spec}}{\text{Characteristic length}}\right) = \sin^{-1}\left(\frac{0.2}{22}\right) = 0.5209^\circ
\]

This gives an upper limit of \(0.5209^\circ\) and a lower limit of \(-0.5209^\circ\). Calculating the \(z\)-score:

\[
z = \frac{x - \text{mean}}{\sigma} = \frac{0.5209^\circ - 0^\circ}{0.2365^\circ} = 2.2029
\]

where \(\sigma = \frac{0.7094^\circ}{3}\), the RSS method predicts approximately 27,600 rejects per 1 million assemblies.
Chapter 6 Example Problem: Quick Return Mechanism

6.1 The Problem

An offset slider crank mechanism is shown below in its two extreme positions (figure 6.1).

![Quick return mechanism diagram](image)

Figure 6.1 Quick return mechanism.

As the crank, a, rotates at a constant speed, the connecting rod, b, moves the slider back and forth on the track. The connecting rod pulls the slider from 1 to 2 on the track, through its power stroke, as the crank rotates counterclockwise from position 1 to position 2. The quicker return stroke occurs as the crank continues to rotate past position 2 and back to 1. Such reciprocating mechanisms are often used in shavers, clippers, feed mechanisms, etc. They are sometimes called quick return mechanisms because the offset, h, causes the forward and return strokes to be unequal. The long forward stroke allows smooth transmission of power. The unloaded return stroke is shortened for efficiency. The stroke length is the critical dimension in the example. In this example the stroke length was assigned an assembly specification of 0.4575 ± 0.05 (figure 6.2).
Figure 6.2 Quick return mechanism assembly specification.

Given the tolerances for the components, an assembly variation for the stroke length needs to be calculated and compared to the assembly specification to determine functionality.

6.2 The Model

The following figure 6.3 shows the vector loops and variables that will be used.

![Vector Loops Diagram]

**Independent Dimensions**
- \( a = 0.1831 \pm 0.005 \)
- \( b = 1.0669 \pm 0.02 \)
- \( h = 0.625 \pm 0.01 \)

**Kinematic Dimensions**
- \( \phi_1 = 30^\circ \pm ? \)
- \( \phi_2 = 120^\circ \pm ? \)
- \( \phi_3 = 45^\circ \pm ? \)
- \( \phi_4 = 45 \pm ? \)
- \( u_1 = 1.0825 \pm ? \)
- \( u_2 = 0.625 \pm ? \)

Figure 6.3 Vector loop for one-way clutch.
Scalar sensitivities are determined using direction cosines, coordinates, and relative rotation directions. The direction cosine method was used for the variables that have translational variations, and the coordinate method was used for the variables that have rotational variations. Figure 6.4 shows the coordinates of the points at which rotational variations occur. Tables 4.1 - 4.3 show the resulting translational scalar sensitivities for the open and the closed loops. There are three tables for these values because some of the vectors change directions in different loops. Table 4.4 shows the resulting rotational scalar sensitivities for all the loops.

Table 6.1 Translational Scalar Sensitivities for Open Loop 1.

<table>
<thead>
<tr>
<th>Vector</th>
<th>Nominal Global Angle ($\theta_i$)</th>
<th>Cos($\theta_i$)</th>
<th>Sin($\theta_i$)</th>
<th>Independent / Form or Dependent</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_1$</td>
<td>180°</td>
<td>-1</td>
<td>0</td>
<td>Dependent</td>
</tr>
<tr>
<td>$b$</td>
<td>30°</td>
<td>0.86603</td>
<td>0.5</td>
<td>Independent</td>
</tr>
<tr>
<td>$a$</td>
<td>30°</td>
<td>0.86603</td>
<td>0.5</td>
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</tr>
<tr>
<td>$h$</td>
<td>-90°</td>
<td>0</td>
<td>-1</td>
<td>Independent</td>
</tr>
</tbody>
</table>

Table 6.2 Translational Scalar Sensitivities for Open Loop 2.

<table>
<thead>
<tr>
<th>Vector</th>
<th>Nominal Global Angle ($\theta_i$)</th>
<th>Cos($\theta_i$)</th>
<th>Sin($\theta_i$)</th>
<th>Independent / Form or Dependent</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_2$</td>
<td>180°</td>
<td>-1</td>
<td>0</td>
<td>Dependent</td>
</tr>
<tr>
<td>$b$</td>
<td>45°</td>
<td>0.70711</td>
<td>0.70711</td>
<td>Independent</td>
</tr>
<tr>
<td>$a$</td>
<td>-45°</td>
<td>-0.70711</td>
<td>-0.70711</td>
<td>Independent</td>
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<tr>
<td>$h$</td>
<td>-90°</td>
<td>0</td>
<td>-1</td>
<td>Independent</td>
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</table>

Table 6.3 Translational Scalar Sensitivities for Open Loop 2.

<table>
<thead>
<tr>
<th>Vector</th>
<th>Nominal Global Angle ($\theta_i$)</th>
<th>Cos($\theta_i$)</th>
<th>Sin($\theta_i$)</th>
<th>Independent / Form or Dependent</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_1$</td>
<td>0°</td>
<td>1</td>
<td>0</td>
<td>Dependent</td>
</tr>
<tr>
<td>$u_2$</td>
<td>180°</td>
<td>-1</td>
<td>0</td>
<td>Dependent</td>
</tr>
</tbody>
</table>
6.3 Sample Analysis Procedure

The sensitivities and variations could be combined into a system of equations and solved for the kinematic variations, and we could add the variations of $u_1$ and $u_2$ to get a resultant variation for the stroke length. But this example is a special case; it is a problem where we are comparing the same components arranged in two different configurations. Specifically, the worst case variation for $u_1$ cannot occur when the variation of $u_2$ is at its worst case. For example, the maximum stroke would occur when $u_1$ is maximum and $u_2$ is minimum. However, because the links $a$ and $b$ are really the same links for both configurations, these two conditions cannot occur simultaneously. If the crank and connecting rod are at their maximum lengths, and the offset is at its minimum length, $u_1$ will be at its maximum variation. For $u_2$ to be at its minimum variation the connecting rod and offset need to be at their maximum lengths, and the crank needs to be at its minimum length.
By combining the sensitivities and variations into a system of equations and solving for the variations in the open loop the problem just explained was overcome (see Appendix C for all matrices). The dimensional sensitivity matrix for the assembly, $[SD]$, calculated from $[C]-[D][B]^{-1}[A]$, combines all of the pertinent scalar sensitivities before applying Worst Case or Root Sum Square tolerance analysis to the component variations.

$$\{dh_{OL}\} = [[C]-[D][B]^{-1}[A]]\{dx\} \quad \text{Equ. 5.1}$$

The open loop variations were determined using worst case and Root Sum Square analyses:

$$\begin{align*}
&\text{WC} & \text{RSS} \\
dx_{OL} &= 0.0223, &= 0.0145
\end{align*}$$

When compared to the assembly specification of 0.05, the Worst Case and RSS analyses estimates that there should be less than 1 reject per million assemblies manufactured.
Chapter 7 Conclusions and Recommendations

7.1 Contributions

The major contributions of this thesis are:

1. Development of a comprehensive set of assembly tolerance specifications for mechanical assemblies.
   a. Definition of three distinct dimensional assembly specifications and five geometric form assembly specifications.

<table>
<thead>
<tr>
<th>Dimensional</th>
<th>Geometric Form</th>
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</thead>
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<tr>
<td>Length</td>
<td>Perpendicularity</td>
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<tr>
<td>Angle</td>
<td>Parallelism</td>
</tr>
<tr>
<td>Gap</td>
<td>Concentricity</td>
</tr>
<tr>
<td></td>
<td>True Position</td>
</tr>
<tr>
<td></td>
<td>Total Runout</td>
</tr>
</tbody>
</table>

   b. The system of tolerances are patterned after ANSI Y14.5M-1982 tolerance specifications by which variations in surfaces or features are controlled by setting tolerance limits.

   c. Each assembly specification relates a surface feature on one component of an assembly to a datum surface on another component of the assembly.

   d. It is applicable to a wide variety of assembly conditions, just like ANSI Y14.5M-1982, which covers a broad range of geometric variations with only 13 feature controls. While the ANSI standard is limited to the relationship between two surfaces on the same part, or at most two mating parts, the new development for assemblies is not.

2. Development of analytical models for each assembly tolerance specification.
   a. Open and closed loop vector models were derived for both linear and angular assembly specs.

   b. A generalized format for the linearized assembly equations for open loops was derived, which includes dimensional, kinematic, and form variations in mechanical assemblies.

   c. Solution procedures have been extended to this more general case.

   d. Worst Case or statistical solutions can be determined, while the ANSI standard is not a statistical approach.

3. Clarification of several assembly tolerance modeling issues.
Chapter 7 Conclusions and Recommendations

4. Testing of CATS 2-D and identifying numerous errors.

ANSI Y14.5M-1982 standards were set up as a comprehensive system for geometric dimensioning and tolerancing of components. Since a parallel has been drawn between component and assembly specifications, the work done here appears to give a comprehensive approach for defining and analyzing assembly specifications. The general method for determining scalar rotational sensitivities has been implemented and tested in CATS, and the mathematical methods developed for geometric form assembly specifications are ready to be implemented into CATS. This will facilitate the ease of applying the concepts of assembly variation analysis and specification in engineering design and manufacturing systems analysis. This will help engineers and manufacturers understand each other's requirements, which will increase the speed and reduce the cost at which products can be developed and produced.

7.2 Recommendations

The work done in this thesis was restricted to 2-D assembly analysis. Although many 3-D assemblies can be modeled with 2-D vector loops, a 3-D system for assembly specification is needed. It is recommended that the system of specifications be extended to 3-D assemblies. A few other problem became apparent while working on this thesis, and will be explained in the following sections.

7.2.1 Assembly variation that occurs when normal kinematic adjustments are restricted.

The analysis done in this thesis assumed all the variations were allowed their full range of motion, and mating parts were assumed to be able to adjust without restrictions. Some cases exist where restrictions must be applied, so some variations will have less of an effect than currently is being assumed. For the assembly in figure 7.1, the centerline of the plunger has an angular variation with respect to the centerline of the base.
The kinematic adjustment the plunger makes is unrestricted when it rotates counterclockwise, but penetrates the base when it rotates clockwise (figure 7.2).

The plunger is restricted because the location of the joint between the plunger and the base is fixed. If the joint were allowed to shift to the outer corner of the base when the plunger rotated clockwise there would be no interference with the base. The arm would shift outward due to the rotation of the wedge on the end of the plunger (figure 7.3).

This problem should be looked at more carefully to better predict the variation that will occur in an assembly.
7.2.3 Extend analysis to allow for variations that do not propagate through assemblies.

Rotational variations are analyzed as if each variation propagates through the entire assembly. If a single part in the assembly has more than one feature tolerance associated with the same datum the analysis may be performed incorrectly. If the features are associated with datums which are the adjoining vectors in the loop, two independent rotational variations propagate (figure 7.4).

![Figure 7.4 Form tolerances that propagate.](image)

If two rotational features are associated with the same datum (figure 7.5) the first rotation, \( \alpha_1 \), will not propagate past \( \alpha_2 \), since \( \alpha_2 \) is not measured relative to Datum B. The variation farthest away from the datum, \( \alpha_2 \), should be used when determining rotational variation, but both need to be used when determining translational variation.

![Figure 7.5 Form tolerances that do not propagate.](image)
REFERENCES


3. Gao, Jinsong (1992), Ph.D. Candidate Brigham Young University.


APPENDIX
### A. Geometric Characteristic Symbols.

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<tr>
<th>Geometric Characteristic</th>
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## B. Remote Positioner.

Spread sheet used to check scalar sensitivities for the remote positioner problem.

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<th>Scalar</th>
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Matrices used in solving the remote positioner example problem.

The matrices are made up of values found in tables 5.1 and 5.2.

Independent scalar sensitivity matrix for closed loops:

\[
[A] = \begin{bmatrix}
\theta_1 & a & b & c & d & \theta_2 & e & \theta_3 & f & g & i & \theta_4 & \theta_5 & j \\
x_1 & 0 & -1 & -0.5038 & 1 & 0.5038 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
y_1 & 0 & 0 & -0.8638 & 0 & 0.8638 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\theta_1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.7356 & 0 & 0.7356 & 0 & 0 \\
y_2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -0.6775 & 1 & 0.6775 & 0 & 0 \\
\theta_2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\
\end{bmatrix}
\]

Dependent scalar sensitivity matrix for closed loops:

\[
[B] = \begin{bmatrix}
\phi_1 & \phi_2 & \phi_3 & \phi_4 & \phi_5 & \phi_6 \\
x_1 & 0 & -8.984 & -8.984 & 0 & 0 & 0 \\
y_1 & 0 & 27.239 & 5.2392 & 0 & 0 & 0 \\
\theta_1 & 1 & 1 & 1 & 0 & 0 & 0 \\
x_2 & 0 & 0 & 0 & 49 & 58.039 & 8.7393 \\
y_2 & 0 & 0 & 0 & 0 & -9.489 & -9.489 \\
\theta_2 & 0 & -1 & 0 & -1 & -1 & -1 \\
\end{bmatrix}
\]

Care must be taken when constructing the [B] matrix because the rotation of \(\phi_2\) is positive in loop 1 and negative in loop 2.

Part tolerances:

\[
\{dx\} = \begin{bmatrix}
\theta_1 \\
a \\
b \\
c \\
d \\
e \\
f \\
g \\
i \\
\theta_4 \\
\theta_5 \\
j \\
\end{bmatrix} = \begin{bmatrix}
0.5^* \\
0.005 \\
0.005 \\
0.005 \\
0.005 \\
0.005 \\
0.005 \\
0.005 \\
0.005 \\
0.005 \\
0.005 \\
0.005 \\
\end{bmatrix}
\]
Each of the open loops will have a $[C]$ and a $[D]$ matrix containing the scalar sensitivities associated with them.

$$
\begin{bmatrix}
\begin{array}{cccccccccccc}
\theta_1 & a & b & c & d & \theta_2 & e & \theta_3 & f & g & i & \theta_4 & \theta_5 & j
\end{array}
\end{bmatrix}
$$

$$
\begin{bmatrix}
\begin{array}{cccccccccccc}
\theta_1 & a & b & c & d & \theta_2 & e & \theta_3 & f & g & i & \theta_4 & \theta_5 & j
\end{array}
\end{bmatrix}
$$

$$
[C_1] = \begin{bmatrix}
58.2839 & -1 & -0.5038 & 0 & 0 & 0 & 0 & 0 & 49.3 & 0 & 0 & 0 & 0 & 0 & 1 \\
-5.2392 & 0 & -0.8638 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 22 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
$$

$$
[C_2] = \begin{bmatrix}
\begin{array}{cccccccccccc}
\theta_1 & a & b & c & d & \theta_2 & e & \theta_3 & f & g & i & \theta_4 & \theta_5 & j
\end{array}
\end{bmatrix}
$$

$$
[D_1] = [D_2] = \begin{bmatrix}
\begin{array}{cccccccccccc}
\phi_1 & \phi_2 & \phi_3 & \phi_4 & \phi_5 & \phi_6
\end{array}
\end{bmatrix}
$$

$$
\begin{bmatrix}
\begin{array}{cccccccccccc}
\begin{array}{cccccccccccc}
\theta_1 & a & b & c & d & \theta_2 & e & \theta_3 & f & g & i & \theta_4 & \theta_5 & j
\end{array}
\end{array}
\end{bmatrix}
$$

The dimensional sensitivity matrices using closed loop 1, $[SD_1]$, and closed loop 2, $[SD_2]$, are calculated using standard linear algebra methods:

$$
[SD_1] = \begin{bmatrix}
\begin{array}{cccccccccccc}
\theta_1 & a & b & c & d & \theta_2 & e & \theta_3 & f & g & i & \theta_4 & \theta_5 & j
\end{array}
\end{bmatrix}
$$

$$
[SD_2] = \begin{bmatrix}
\begin{array}{cccccccccccc}
\theta_1 & a & b & c & d & \theta_2 & e & \theta_3 & f & g & i & \theta_4 & \theta_5 & j
\end{array}
\end{bmatrix}
$$
C. Quick Return Mechanism.

Spread sheet used to check quick return mechanism problem.

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<th>Global angle</th>
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<th>45°</th>
<th>-135°</th>
<th>-90°</th>
<th>180°</th>
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