# General 2-D Tolerance Analysis of Mechanical Assemblies with Small Kinematic Adjustments

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## Abstract

Assembly tolerance analysis is a key element in industry for improving product quality and reducing overall cost. It provides a quantitative design tool for predicting the effects of manufacturing variation on performance and cost. It promotes concurrent engineering by bringing engineering requirements and manufacturing requirements together in a common model.

A new method, called the Direct Linearization Method (DLM), is presented for tolerance analysis of 2-D mechanical assemblies which generalizes vector loop-based models to include small kinematic adjustments. It has a significant advantage over traditional tolerance analysis methods in that it does not require an explicit function to describe the relationship between the resultant assembly dimension(s) and manufactured component dimensions. Such an explicit assembly function may be difficult or impossible to obtain for complex 2-D assemblies.

The DLM method is a convenient design tool. The models are constructed of common engineering elements: vector chains, kinematic joints, assembly datums, dimensional tolerances, geometric feature tolerances and assembly tolerance limits. It is well suited for integration with a commercial CAD system as a graphical front end. It is not computationally intensive, so it is ideally suited for iterative design.

A general formulation is derived, detailed modeling and analysis procedures are outlined and the method is applied to two example problems.

# 1. Introduction

An important consideration in product design is the assignment of tolerances to individual component dimensions so the product can be produced economically and function properly. The designer may assign relatively tight tolerances to each part to ensure that the product will perform correctly, but this will generally drive manufacturing cost higher. Relaxing tolerances on each component, on the other hand, reduces costs, but can result in unacceptable loss of quality and high scrap rate, leading to customer dissatisfaction. These conflicting goals point out the need in industry for methods to rationally assign tolerances to products so that customers can be provided with high quality products at competitive market prices.

Clearly, a tool to evaluate tolerance requirements and effects would be most useful in the design stage of a product. To be useful in design, it should include the following characteristics:

1. Bring manufacturing considerations into the design stage by predicting the effects of manufacturing variations on engineering requirements.

2. Provide built-in statistical tools for predicting tolerance stack-up and percent rejects in assemblies.

3. Be capable of performing 2-D and 3-D tolerance stack-up analyses.

4. Be computationally efficient, to permit design iteration and design optimization.

5. Use a generalized and comprehensive approach, similar to finite element analysis, where a few basic elements are capable of describing a wide variety of assembly applications and engineering tolerance requirements.

6. Incorporate a systematic modeling procedure that is readily accepted by engineering designers.

7. Be easily integrated with commercial CAD systems, so geometric, dimensional and tolerance data may be extracted directly from the CAD database.

8. Use a graphical interface for assembly tolerance model creation and graphical presentation of results.

To illustrate the problems associated with 2-D tolerance analysis, consider the simple assembly shown in figure 1, as described by Fortini [1967]. It is a drawing of a one-way mechanical clutch. This is a common device used to transmit rotary motion in only one direction. When the outer ring of the clutch is rotated clockwise, the rollers wedge between the ring and hub, locking the two so they rotate together. In the reverse direction, the rollers just slip, so the hub does not turn. The pressure angle $\mathbf{\Phi_1}$ between the two contact points is critical to the proper operation of the clutch. If $\mathbf{\Phi_1}$ is too large, the clutch will not lock; if it is too small the clutch will not unlock.

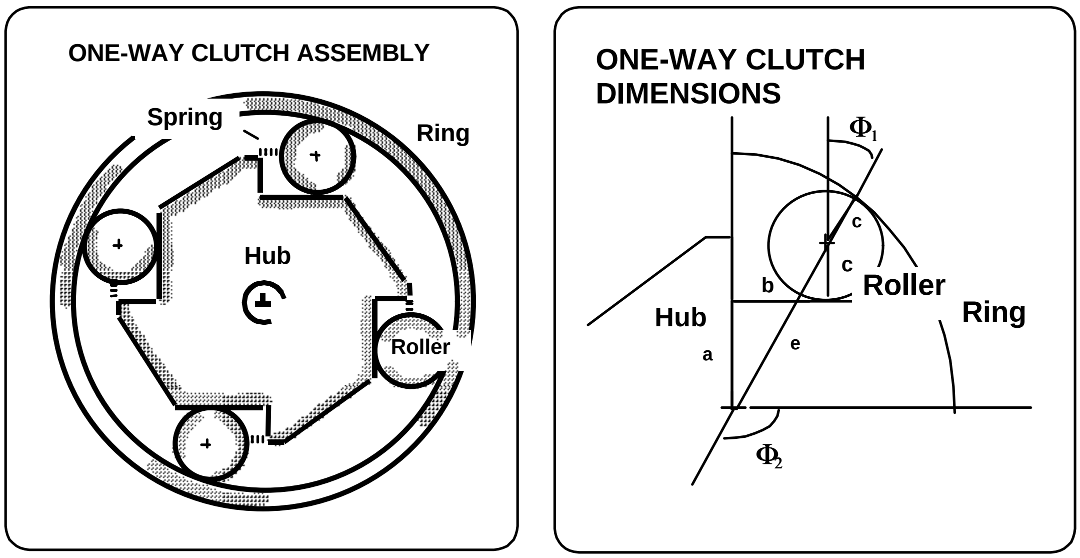


Figure 1. One-way clutch assembly and its relevant dimensions

The primary objective of performing a tolerance analysis on the clutch is to determine how much the angle $\mathbf{\Phi_1}$ is expected to vary due to manufacturing variations in the clutch component dimensions. The independent manufacturing variables are the hub dimension **a**, the cylinder radius **c**, and the ring radius **e**. The distance **b** and angle $\mathbf{\Phi_1}$ are not dimensioned. They are assembly resultants which are determined by the sizes of **a**, **c** and **e** when the parts are assembled. By trigonometry, the dependent assembly resultants, distance **b** and angle $\mathbf{\Phi_1}$, can be expressed as explicit functions of **a**, **c** and **e**.

$$\mathbf{\Phi_1} = \cos^{-1}\left(\frac{\mathbf{a} + \mathbf{c}}{\mathbf{e} - \mathbf{c}}\right) \qquad \mathbf{b} = \sqrt{(\mathbf{e} - \mathbf{c})^2 - (\mathbf{a} + \mathbf{c})^2} \qquad (1)$$

The expression for angle $\mathbf{\Phi_1}$ may be analyzed statistically to estimate quantitatively the resulting variation in $\mathbf{\Phi_1}$ in terms the specified tolerances for **a**, **c** and **e**. If performance requirements are used to set engineering limits on the size of $\mathbf{\Phi_1}$, the quality level and percent rejects may also be predicted.

When an explicit function of the assembly resultant is available, such as $\mathbf{\Phi_1}$ in equation (1), several methods are available for performing a statistical tolerance analysis. These include:

1. Linearization of the assembly function using Taylor series expansion,
2. Method of system moments,
3. Quadrature,
4. Monte Carlo simulation,
5. Reliability index,
6. Taguchi method.

The next section will briefly review these methods.

Establishing explicit assembly functions, such as equation (1), to describe assembly kinematic adjustments, places a heavy burden on the designer. For a general mechanical assembly, this relationship may be difficult or impossible to obtain. Figure 2 shows a geometric block assembly. The resultant dimension $\mathbf{U_1}$ is very difficult to express explicitly as a function of only the independent component dimensions **a**, **b**, **c**, **d**, **e** and **f**. It is very difficult to define such explicit assembly functions in a generalized manner for "real-life" mechanical assemblies. This difficulty makes the use of explicit functions impractical in a CAD-based system intended for use by mechanical designers.

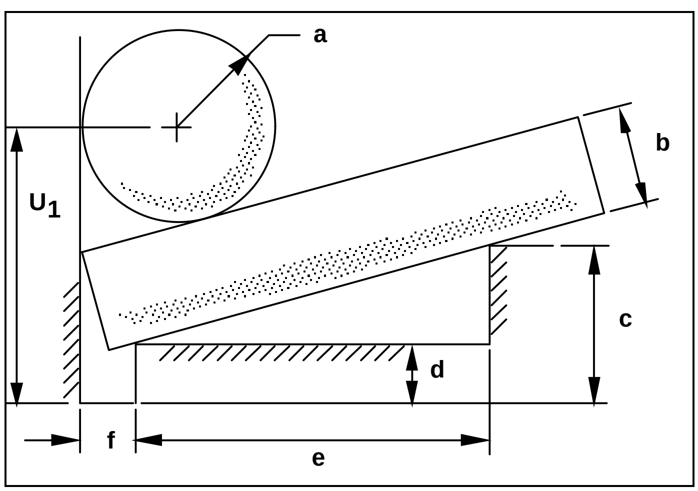


Figure 2. Simple geometric block assembly

The approach described in this paper solves the problem mentioned above by using implicit assembly functions with a vector-loop-based kinematic assembly model, so that less user intervention is needed for computer-aided tolerance analysis of any mechanical assemblies. The next section reviews the principal methods that have been used for tolerance analysis. The following sections introduce the concepts of variation sources and assembly kinematics. The formulation of the DLM assembly tolerance analysis method is then presented, followed by specific examples.

## 2. Methods Available for Tolerance Analysis

This section will briefly review the methods available for nonlinear tolerance analysis when an explicit assembly function is provided which relates the resultant variables of interest to the contributing variables or dimensions in an assembly. The purpose of the review is to provide background for a discussion of a generalized method for treating implicit functions.

### 2.1 Linearization Method

The linearization method is based on a first order Taylor series expansion of the assembly function, such as equation (1). Then the variation $\Delta\mathbf{\Phi_1}$ may be estimated by a worst case or statistical model for tolerance accumulation [Cox 1986, Shapiro & Gross 1981].

$$\Delta\mathbf{\Phi_1} = \left|\frac{\partial \mathbf{F_1}}{\partial \mathbf{a}}\right| \mathrm{tol_a} + \left|\frac{\partial \mathbf{F_1}}{\partial \mathbf{c}}\right| \mathrm{tol_c} + \left|\frac{\partial \mathbf{F_1}}{\partial \mathbf{e}}\right| \mathrm{tol_e} \qquad \text{(Worst Case)} \quad (2)$$

$$\Delta\mathbf{\Phi_1} = \sqrt{\left(\frac{\partial \mathbf{F_1}}{\partial \mathbf{a}} tol_a\right)^2 + \left(\frac{\partial \mathbf{F_1}}{\partial \mathbf{c}} tol_c\right)^2 + \left(\frac{\partial \mathbf{F_1}}{\partial \mathbf{e}} tol_e\right)^2} \qquad \text{(Statistical)} \quad (3)$$

The derivatives of $\mathbf{\Phi_1}$ with respect to each of the independent variables **a**, **c** and **e** are called the "tolerance sensitivities", and are essential to the models for accumulation, hence, the need for an explicit function is apparent.

### 2.2 System Moments

System moments is a statistical method for expressing assembly variation in terms of the moments of the statistical distributions of the components in the assembly. The first four moments describe the mean, variance, skewness and kurtosis of the distribution, respectively. A common procedure is to determine the first four moments of the assembly variable and use these to match a distribution that can be used to describe system performance [Evans 1975a, 1975b, Cox 1979, 1986, Shapiro & Gross 1981].

Moments are obtained from a Taylor's series expansion of the assembly function $\mathbf{\Phi_1}(x_i)$ about the mean, retaining higher order derivative terms, as shown in equation 4:

$$E[m_k] = E\Big[\sum_{i=1}^{n} \frac{\partial \mathbf{F_1}}{\partial x_i}(\mathbf{F_1}(x_i) \;-\mathbf{F_1}(m_i + \sum_{i=1}^{n} \frac{\partial^2 \mathbf{F_1}}{\partial x_{i^2}}(\mathbf{F_1}(x_i) \;-\mathbf{F_1}(m_i$$

$$+ \sum_{i<j}^{n-1}\sum_{j=1}^{n} \frac{\partial^2 \mathbf{F_1}}{\partial x_i \partial x_j}(\mathbf{F_1}(x_i) - \mathbf{F_1}(m_i(\mathbf{F_1}(x_j) \;-\mathbf{F_1}(m_j + \ldots\Big]^k \qquad (4)$$

where $m_k$ is the kth moment, E is the expected value operator, $x_i$ are the variables **a**, **c, and e**, and $\mu_i$ are their mean values. Expanding the truncated series to the third and fourth power yields extremely lengthy expressions for the third and fourth moments.

Clearly, this method also relies on an explicit assembly function.

### 2.3 Quadrature

The basic idea of quadrature is to estimate the moments of the probability density function of the assembly variable by numerical integration of a moment generating function, as shown in equation 5:

$$E[m_k] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[\phi_1(a,c,e) - \phi_1(u_a,u_c,u_e)\right]^k \; w(a)w(c)w(e)da\,dc\,de \qquad (5)$$

where $m_k$ is the kth moment of the assembly distribution, w(a), w(c) and w(e) are the probability density functions for the independent variables **a**, **c** and **e**, and $\mu_a$ , $\mu_c$ and $\mu_e$ are their mean values. Engineering limits are then applied to the resulting assembly distribution to estimate the statistical performance of the system [Evans 1967, 1971, 1972].

### 2.4 Reliability Index

The Hasofer-Lind Reliability Index, also called Second Moment Reliability Index, was originally developed for structural engineering applications [Hasofer & Lind 1974, Ditlevsen 1979a, 1979b]. This sophisticated method has been applied to mechanical tolerance analysis [Parkinson 1978, 1982, 1983, Lee & Woo 1990]. The reliability index may be used to approximate the distance of each engineering limit from the mean of the assembly, and estimate the percent rejects. It requires only the means and covariances of the independent variables, which assumes that all the independent variables are normally distributed and independent.

### 2.5 Taguchi Method

The general idea of the Taguchi method is to use fractional factorial or orthogonal array experiments to estimate the assembly variation due to component variations. It may further be applied to find the nominal dimensions and tolerances which minimize a specified “loss function”. The Taguchi method is applicable to both explicit and implicit assembly functions [Taguchi 1978].

### 2.6 Monte Carlo Simulations

The Monte Carlo simulation method evaluates individual assemblies using a random number generator to select values for each manufactured dimension, based on the type of statistical distribution assigned by the designer or determined from production data. These dimensions are combined through the assembly function to determine the value of the assembly variable for each simulated assembly. This set of values is then used to compute the first four moments of the assembly variable. Finally, the moments may be used to determine the system behavior of the assembly, such as the mean, standard deviation, and percentage of assemblies which fall outside the design specifications [Sitko 1991, Fuscaldo 1991, Craig 1989].

An explicit assembly function is required to permit substitution of random sets of component dimensions and compute the change in assembly variables for each assembly.

## 3. Variation Sources in Assemblies

In order to create a generalized approach for generating implicit assembly functions, the sources of variation in an assembly must be identified and categorized. With these categories in place, an engineer can use them to systematically create a model that can be used to derive the implicit functions.

There are three main sources of variation in a mechanical assembly: 1) dimensional variation, 2) geometric feature variation and 3) variation due to small kinematic adjustments which occur at assembly time. The first two are the result of the natural variations in manufacturing processes and the third is from assembly processes and procedures.

Figure 3 shows sample dimensional variations on a component. Such variations are inevitable due to fluctuations of machining conditions, such as tool wear, fixture errors, set up errors, material property variations, temperature, worker skill, etc. The designer usually specifies limits for each dimension. If the manufactured dimension falls within the specified limits, it is considered acceptable. Since this variation will affect the performance of the assembled product, it must be carefully controlled.

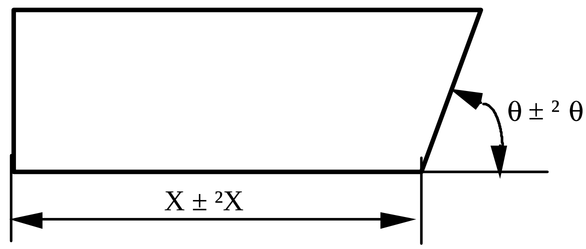


Figure 3. Example of dimensional variations

Geometric feature variations are defined by the ANSI Y14.5M-1982 standard [ASME 1982]. These definitions provide additional tolerance constraints on shape, orientation, and location of produced components. For example, a geometric feature tolerance may be used to limit the flatness of a surface, or the perpendicularity of one surface on a part relative to established datums, as shown in Figure 4.

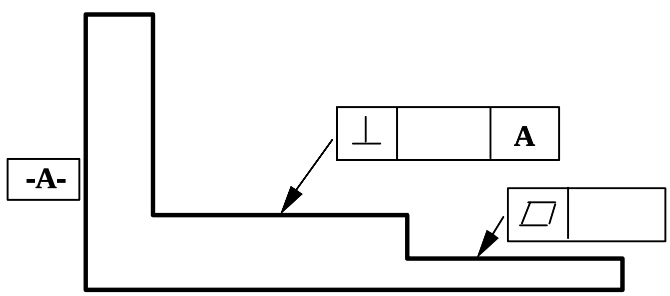


Figure 4. Example of geometric feature variation limits.

In an assembly, geometric feature variations accumulate and propagate similar to dimensional variations. Although generally smaller than dimensional variations, they may be significant in some cases, resulting from rigid body effects [Ward 1992]. A complete tolerance model of mechanical assemblies should therefore include geometric feature tolerances.

Kinematic variations are small adjustments between mating parts which occur at assembly time in response to the dimensional variations and geometric feature variations of the components in an assembly. For example, if the roller in the clutch assembly is produced undersized, as shown in figure 5, the points of contact with the hub and ring will change, causing kinematic variables **b** and $\mathbf{\Phi_1}$ to increase.

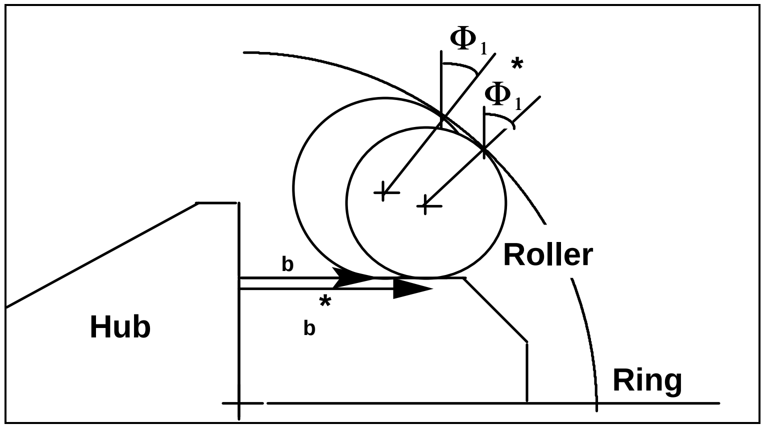


Figure 5. Example of kinematic or assembly variations due to a change in the roller size.

Usually, limiting values of kinematic variations are not marked on the mechanical drawing, but critical performance variables, such as a clearance or a location, may appear as assembly specifications. The task for the designer is to assign tolerances to each component in the assembly so that each assembly specification is met.

It is the kinematic variations which result in implicit assembly functions. Current tolerance analysis practices fail to account for this significant variation source. In a comprehensive assembly tolerance analysis model, all three variations should be included. If any of the three is overlooked or ignored, it can result in significant error. Only when a complete model is constructed, can the designer accurately estimate the resultant assembly features or kinematic variations in an assembly.

## 4. Assembly Kinematics

Since an assembly must adjust to accommodate the three types of variation, a model of the assembly must account for kinematics. The kinematics present in a tolerance analysis model of an assembly is different from the traditional mechanism kinematics. The input and output of the traditional mechanism are large displacements of the corresponding components, such as the rotation of the input and output cranks of a four-bar linkage.

The linkage is composed of rigid bodies, so all the component dimensions remain constant, or fixed at their nominal values.

In contrast to this, the kinematic inputs of an assembly tolerance analysis model are small variations of the component dimensions around their nominal values and the outputs are the variations of assembly features, including clearances and fits critical to performance, as well as small kinematic adjustments between components.

The kinematic adjustments have a similar meaning to kinematic degrees of freedom, but the input motions do not refer to displacements of a mechanism. They actually represent differences from the nominal dimension from one assembly to the next.

The kinematic assembly equations describe constraints on the interaction between mating component parts. These constraints also serve as functions by which assembly variations may be studied. Since the assembly model is similar to a classical kinematic mechanism model, the analysis methods developed for mechanism kinematics can be applied to assembly variation analysis.

## 5. Vector-Loop-Based Assembly Models

Using the concepts presented in section 3 and 4, vector-loop-based assembly models use vectors to represent the dimensions in an assembly. Each vector represents either a component dimension or kinematically variable dimension. The vectors are arranged in chains or loops representing those dimensions which "stack" together to determine the resulting assembly dimensions. The other model elements include kinematic joints, datum reference frames, feature datums, assembly tolerance specifications, component tolerances, and geometric feature tolerances (Figure 6).

Kinematic joints describe motion constraints at the points of contact between mating parts. The assembly tolerance specifications are the engineering design limits on those assembly feature variations which are critical to performance. Vector models can provide a broad spectrum of the necessary assembly functions for tolerance analysis.

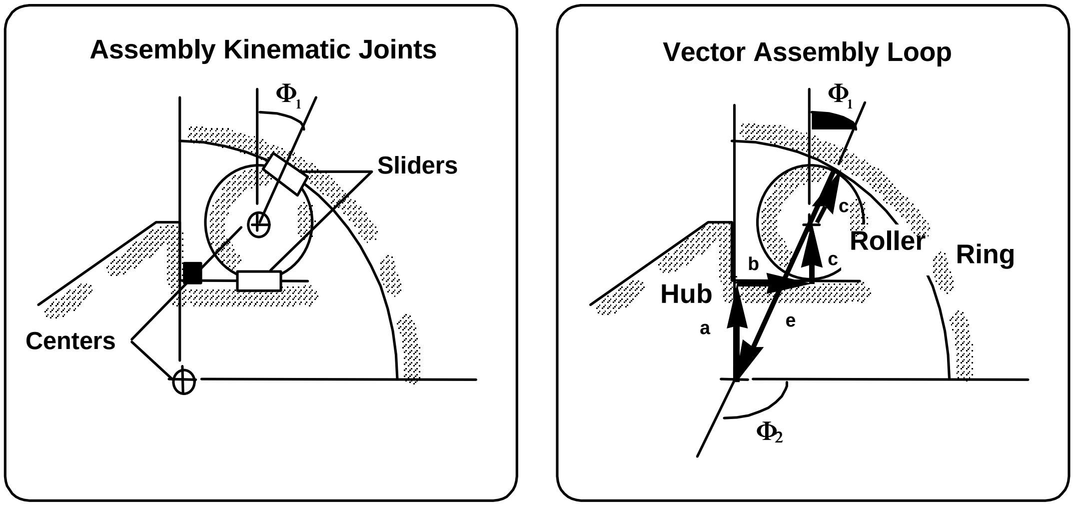


Figure 6  Kinematic joints and vector loop representing the one-way clutch assembly

Marler [1988] and Chun [1988] defined a set of kinematic joint types to accommodate the kinematic variations at the contact points in 2-D assemblies.  Figure 7 shows the joints and datums for 2-D analysis.  Corresponding modeling rules have been developed for correctly representing the kinematic degrees of freedom in an assembly.

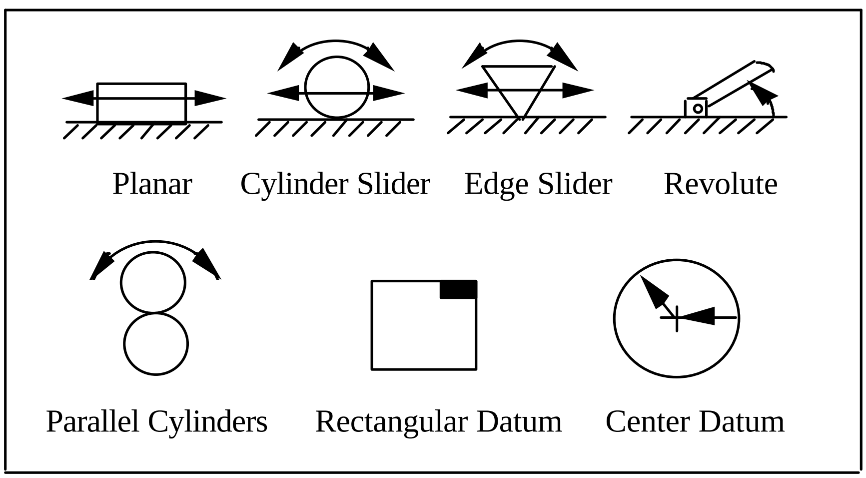


Figure 7.  2-D kinematic joint and datum types

Larsen [1991] and Trego [1993] further developed Chun and Marler's work and automated the procedure of generating vector loop relationships for assemblies.

There are several major advantage of vector models over solid models of assemblies:

1. The geometry is reduced to only those parameters that are required to perform a tolerance analysis.

2. Tolerance sensitivities can be determined in closed form, eliminating a computation-intensive task.

3. Kinematic constraints on relative motions and geometric form, location and orientation variations are readily introduced into vector models.

4. Dimensions and tolerances may be associated with the vectors for those solid modeling systems which do not store this information.

## 6. DLM - Linearization of Implicit Assembly Functions

The Direct Linearization Method for assembly tolerance analysis is based on the first order Taylor's series expansion of the assembly kinematic constraint equations with respect to both the assembly variables and the manufactured variables (component dimensions) in an assembly. Linear algebra is employed to solve the resulting linearized equations for the variations of the assembly variables in terms of the variations of the manufactured components. The resulting explicit expressions may be evaluated by either a worst case or statistical tolerance accumulation model.

### 6.1 Assembly Kinematic Constraint

Figure 8 shows a vector loop model of an assembly in 2-D. Each vector defines the relative rotation and translation from the previous vector. If a vector represents a component dimension, then its variation is the specified component tolerance. If it is a kinematic variable, its variation must be determined by solving the vector equation. A similar interpretation holds for the relative angles. Whether a length or angle is a kinematic variable is determined by the degrees of freedom of the corresponding kinematic joint defined at the points of contact between mating parts.

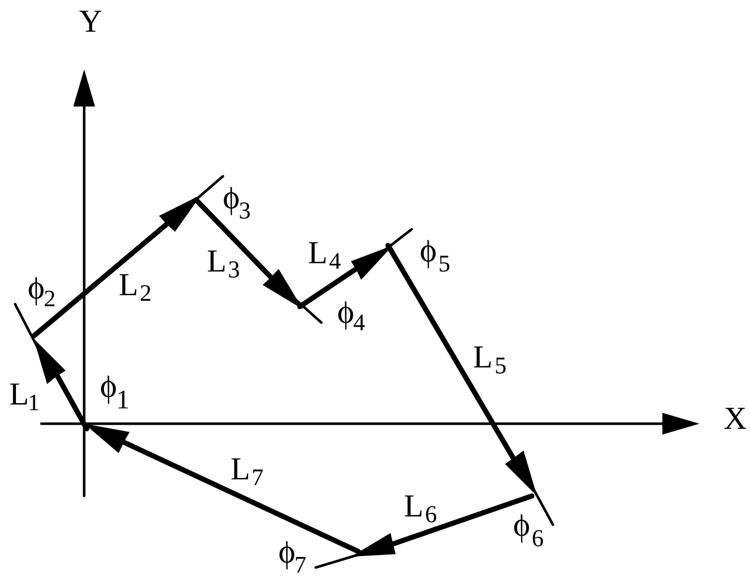


Figure 8  A sample vector-loop-based assembly model

A closed vector loop, such as that shown in figure 8, defines a kinematic closure constraint for the assembly. This means there is some adjustable element in the assembly which always permits closure. Closed loop constraints can readily be expressed as implicit assembly functions.

An open vector loop describes a gap or a stack dimension, corresponding to a critical assembly feature which is the result of the accumulation of component tolerances.

Many assembly applications are described by an implicit system of open and closed loops requiring simultaneous solution. When no adjustable elements are present, no closed loops are required, in which case, open loops may be expressed as explicit assembly functions.

Assembly tolerance limits are determined by performance requirements. Component tolerance limits are determined from process characterization studies, but may have to be modified as a result of tolerance analysis, which reveals how each component variation contributes to the overall assembly variation. Engineering design limits may be placed on any kinematic variation in a closed loop or any assembly feature variation defined by an open loop. By comparing the computed variations to the specified limits, the percent rejects and assembly quality levels may be estimated.

By summing the vector components in the global x and y directions and summing the relative rotations, a vector loop produces three scalar equations, each summing to zero, as shown in equations 6, 7 and 8.

$$H_x = \sum_{i=1}^{n} L_i \cos\left(\sum_{j=1}^{i} \phi_j\right) = 0 \qquad (6)$$

$$H_y = \sum_{i=1}^{n} L_i \sin\left(\sum_{j=1}^{i} \phi_j\right) = 0 \qquad (7)$$

$$H_\phi = \sum_{i=1}^{n} \phi_j = 0 \text{ or } 360° \qquad (8)$$

It is significant that each vector direction, represented by the arguments of the sine and cosine functions in the above equations, is expressed as the sum of the relative angles of all the vectors preceding it in the loop. Both the manufactured and kinematic angles are relative angles. This allows rotational variations to propagate realistically through an assembly, producing rigid body rotations of stacked mating parts. This effect of individual angle variations could not be described if global angles were used in the equations.

### 6.2. Taylor's Expansion of Implicit Assembly Functions

The first order Taylor's series expansion of the closed loop assembly equations can be written in matrix form:

$$\{\Delta H\} = [A]\{\Delta X\} + [B]\{\Delta U\} = \{\Theta\} \qquad (9)$$

where $\{\Delta H\}$: the variations of the clearance or closure relation,
$\{\Delta X\}$: the variations of the manufactured variables,
$\{\Delta U\}$: the variations of the assembly variables,
$[A]$: the first order partial derivatives of the manufactured variables,
$[B]$: the first order partial derivatives of the assembly variables.

Solving equation (9) for $\Delta U$ gives (assuming that [B] is a full-ranked matrix):

$$\{\Delta U\} = -[B]^{-1}[A]\{\Delta X\} \qquad (10)$$

For an open loop assembly constraint, there may also be one or more closed loop assembly constraints which the assembly must satisfy. The strategy for such a system of assembly constraints is to solve the closed loop constraints first, then substitute the solution in the open loop assembly constraint. The variations of the open loop variables may then be evaluated directly. This procedure may be expressed mathematically as follows:

$$\{\Delta V\} = [C]\{\Delta X\} + [D]\{\Delta U\} \tag{11}$$

where $\Delta V$: the variations of the open loop assembly variables,

[C]: the first order derivative matrix of the manufactured variables in the open loop,

[D]: the first order derivative matrix of the assembly variables in the open loop.

If [B] is full-ranked, equation (11) may be written as:

$$\{\Delta V\} = ([C] - [D][B]^{-1}[A])\{\Delta X\} \tag{12}$$

**6.3. Estimation of Kinematic Variations and Assembly Rejects**

The estimation of the kinematic variations can be obtained from equation (10) for the closed loop constraint, or equation (12) for the open loop constraint, by a worst case or statistical tolerance accumulation model.

Worst case:

$$\Delta U_i = \sum_{j=1}^{n} |S_{ij}| \, tol_j \leq T_{ASM} \tag{13}$$

Statistical model:

$$\Delta U_i = \sqrt{\sum_{j=1}^{n} (S_{ij}\, tol_j)^2} \leq T_{ASM} \tag{14}$$

where i = 1,..., n, $tol_j$ is the tolerance of the jth manufactured dimension, $T_{ASM}$ is the design specification for the ith kinematic variable and [S] is the tolerance sensitivity matrix of the assembly constraint.

For closed loop constraints

$$[S] = -[B]^{-1}[A] \tag{15}$$

For open loop constraints

$$[S] = [C] - [D][B]^{-1}[A] \quad (16)$$

The estimation of the assembly rejects is based on the assumption that the resulting sum of component distributions is Normal or Gaussian, which is a reasonable estimate for assemblies of manufactured variables. If all the component tolerances are assumed to represent three standard deviations of the corresponding process, the estimate of the related assembly variation will be three standard deviations. Equation 14 may easily be modified to account for tolerance limits which represent a value other than three standard deviations. The mean and standard deviation of the assembly variable can be used to calculate by either integration or table the assembly rejects for a given production quantity of assemblies.

## 7. Examples

As examples to demonstrate the procedure of applying DLM assembly tolerance analysis method to real assemblies, the one-way clutch assembly and the geometric block assembly are re-examined in greater detail.

### 7.1. Example 1. One-Way Clutch

Figure 6 illustrated the vector-loop-based model of the one-way clutch assembly. Table 1 shows the detailed dimensions for the assembly.

**Table 1 Dimensions of one-way clutch vector loop**

| Part Name | Transformation | Nominal Dimension | Tolerance(±) | Variation |
|---|---|---|---|---|
| Height of hub | Rotation | 90° | | |
| | Translation **a** | 27.645 | 0.0125 | Independent |
| Position of roller | Rotation | -90° | | |
| | Translation **b** | 4.81053 | ? | Kinematic |
| Radius of roller | Rotation | 90° | | |
| | Translation **c** | 11.43 | 0.01 | Independent |
| Radius of roller | Rotation $\phi_1$ | -7.01838° | ? | Kinematic |
| | Translation **c** | 11.43 | 0.01 | Independent |
| Radius of ring | Rotation | 180° | | |
| | Translation **e** | 50.8 | 0.05 | Independent |
| Closing vector | Rotation $\phi_2$ | 97.01838° | ? | Kinematic |
| | Translation | 0 | | |

The question marks in the above table indicate the kinematic variations which must be determined by tolerance analysis.

From Figure 6, the loop equations of the assembly follow naturally as:

$$H_x = \mathbf{b} + \mathbf{c}\cdot\cos(90°+ \phi_1) + \mathbf{e}\cdot\cos(270°+ \phi_1) = 0 \qquad (17)$$

$$H_y = \mathbf{a} + \mathbf{c} + \mathbf{c}\cdot\sin(90°+ \phi_1) + \mathbf{e}\cdot\sin(270°+ \phi_1) = 0 \qquad (18)$$

$$H_\theta = 90° - 90° + 90° + \phi_1 + 180° + \phi_2 = 0 \qquad (19)$$

The known independent variables in this set of equations are **a**, **c**, and **e**. The unknown dependent variables are **b**, $\phi_1$ and $\phi_2$. Examination of the system of equations reveals that they are nonlinear functions of $\phi_1$, which must be solved simultaneously for all three dependent variables. It is not clear how one would apply the tolerance analysis methods described earlier to a system of implicit assembly functions such as this, without first solving symbolically for an explicit function of $\phi_1$ in terms of **a**, **c**, and **e**.

Note that dimension **c** appears twice in equation 18. Since both vectors are produced by the same process, they will both be oversized or undersized simultaneously.

Applying the DLM method, the first order derivative matrices [A], [B] and the sensitivity matrix [S] can be obtained.

$$[\,A\,] = \begin{bmatrix} \frac{\partial H_x}{\partial a} & \frac{\partial H_x}{\partial c} & \frac{\partial H_x}{\partial e} \\ \frac{\partial H_y}{\partial a} & \frac{\partial H_y}{\partial c} & \frac{\partial H_y}{\partial e} \\ \frac{\partial H_q}{\partial a} & \frac{\partial H_q}{\partial c} & \frac{\partial H_q}{\partial e} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0.1222 & -0.1222 \\ 1 & 1.9925 & -0.9925 \\ 0 & 0 & 0 \end{bmatrix} \qquad (21)$$

$$[\,B\,] = \begin{bmatrix} \frac{\partial H_x}{\partial b} & \frac{\partial H_x}{\partial f_1} & \frac{\partial H_x}{\partial f_2} \\ \frac{\partial H_y}{\partial b} & \frac{\partial H_y}{\partial f_1} & \frac{\partial H_y}{\partial f_2} \\ \frac{\partial H_q}{\partial b} & \frac{\partial H_q}{\partial f_1} & \frac{\partial H_q}{\partial f_2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 39.075 & 0 \\ 0 & -4.811 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad (22)$$

$$[\,S\,] = -[\,B\,]^{-1}[\,A\,]$$

$$= \begin{bmatrix} -8.1220 & -16.305 & 8.1833 \\ 0.2079 & 0.4142 & -0.2063 \\ -0.2079 & -0.4142 & 0.2063 \end{bmatrix} \qquad (23)$$

With the sensitivity matrix known, the variations of the kinematic or assembly variables can then be calculated by applying equation (13) or (14).

Worst case:

$$\begin{Bmatrix} \Delta \mathbf{b} \\ \Delta f_1 \\ \Delta f_2 \end{Bmatrix} = \begin{Bmatrix} 0.6737 \\ 0.9772° \\ 0.9772° \end{Bmatrix}$$

Statistical model:

$$\begin{Bmatrix} \Delta \mathbf{b} \\ \Delta f_1 \\ \Delta f_2 \end{Bmatrix} = \begin{Bmatrix} 0.4520 \\ 0.6540° \\ 0.6540° \end{Bmatrix} \qquad (24)$$

In this assembly, dimension $\phi_1$ is the one which has a specified design tolerance since its mean value and variation will affect the performance of the clutch. The design limits for $\phi_1$ are set to be $T_{ASM} = \pm 0.6°$, with a desired quality level of ±3.0 standard deviations.

The number of standard deviations Z to which the design spec corresponds may be calculated from the relation:

$$Z = \frac{T_{ASM}}{\sigma_1} = \frac{0.6}{0.6540} * 3.0 = 2.7523 \qquad (25)$$

This standard deviation number can then be used to estimate the assembly reject rate $\eta$ by either standard normal distribution tables or integration or empirical methods.

$$\text{Reject Rate} = \eta = 0.002959 \quad \text{dpu, or defects per unit} \qquad (26)$$

The assembly rejects for a production run of 1000 assemblies can be estimated by

$$\begin{aligned} \text{Assembly Rejects} &= 2\eta * \text{Number of the Assemblies} \qquad (27) \\ &= 2(0.002959)1000 \\ &= 5.918 \end{aligned}$$

So, there would be about six which would function improperly (three at each design limit).

### 7.2. Example 2. Geometric Block Assembly

The geometric block assembly requires three vector loops to completely describe the assembly relationship, even though it is only a simple three-component assembly. Figure 9 shows the vector loop assembly model. Table 2 gives all the dimensions for the three vector loops.

**Table 2 Dimensions of the geometric block assembly**

| Loop Name | Part Name | Transformation | Nominal Dim | Tolerance(±) |
|---|---|---|---|---|
| Loop 1 | Ground | Rotation | 90° | |
| | | Translation $\mathbf{U_3}$ | 10.0477 | ? |
| | Block | Rotation $\phi_2$ | -74.7243° | ? |
| | | Translation $\mathbf{U_2}$ | 8.6705 | ? |
| | Cylinder | Rotation | 90° | |
| | | Translation **a** | 6.62 | 0.2 |
| | Cylinder | Rotation $\phi_1$ | 74.7243° | ? |
| | | Translation **a** | 6.62 | 0.2 |
| Closing angle | Ground | Rotation | 90° | |
| 90° | | Translation $\mathbf{U_1}$ | 18.7181 | ? |
| Loop 2 | Ground | Rotation | 90° | |
| | | Translation $\mathbf{U_3}$ | 10.0477 | ? |
| | Block | Rotation $\phi_2$+90° | -164.7243° | ? |
| | | Translation **b** | 6.805 | 0.075 |
| | Block | Rotation | 90° | |
| | | Translation $\mathbf{U_4}$ | 2.1894 | ? |
| | Ground | Rotation $\phi_3$ | -105.2761° | ? |
| | | Translation **d** | 4.06 | 0.15 |
| Closing angle | Ground | Rotation | -90° | |
| 180° | | Translation **f** | 3.905 | 0.125 |
| Loop 3 | Ground | Rotation | 90° | |
| | | Translation $\mathbf{U_3}$ | 10.0477 | ? |
| | Block | Rotation $\phi_2$+90° | -164.7243° | ? |
| | | Translation **b** | 6.805 | 0.075 |
| | Block | Rotation | 90° | |
| | | Translation $\mathbf{U_5}$ | 27.2965 | ? |
| | Ground | Rotation $\mathbf{U_4}$ | -105.2761° | ? |
| | | Translation **c** | 10.675 | 0.125 |
| | Ground | Rotation | -90° | |
| | | Translation **e** | 24.22 | 0.35 |

| | | | | |
|---|---|---|---|---|
| Closing angle 180° | Ground | Rotation | 0° | |
| | | Translation **f** | 3.905 | 0.125 |

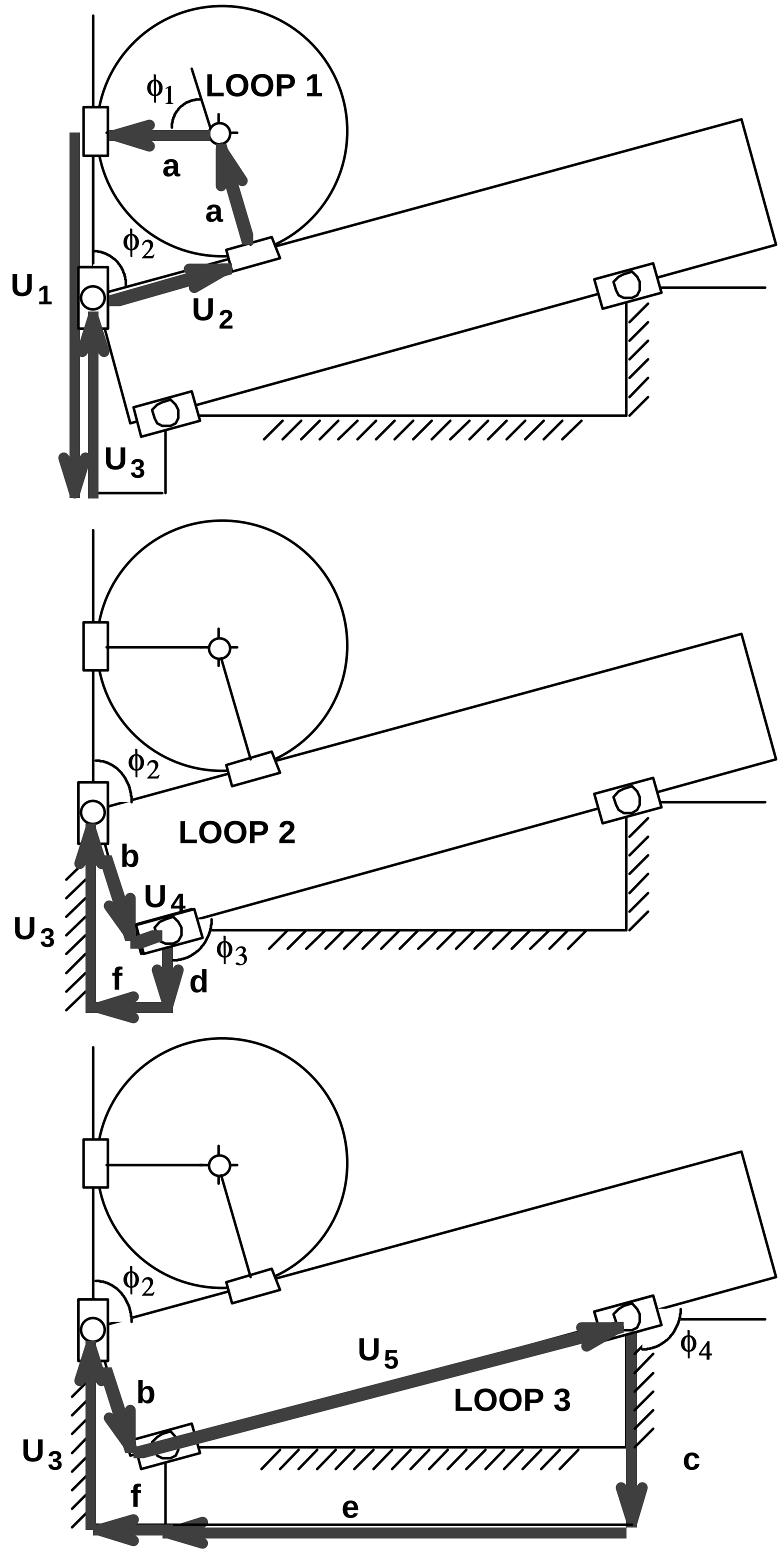

Figure 9  Vector loop modeling of the geometric block assembly

For each vector loop, three equations having the same format as equation (17) to (19) can be obtained.  Therefore, nine equations are required to describe the assembly.

$$[A]=\left[\frac{H_x^i}{\partial X_j},\frac{H_y^i}{\partial X_j},\frac{H_q^i}{\partial X_j}\right]^T$$ the no. of loops; j: the no. of independent variables

$$
\begin{array}{cccccc}
\mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} & \mathbf{e} & \mathbf{f}
\end{array}
$$

$$
=\begin{bmatrix}
-1.2635 & 0 & 0 & 0 & 0 & 0 \\
0.9647 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.2635 & 0 & 0 & 0 & -1 \\
0 & -0.9647 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.2635 & 0 & 0 & -1 & -1 \\
0 & -0.9647 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \qquad (27)
$$

$$[B]=\left[\frac{H_x^i}{\partial U_j},\frac{H_y^i}{\partial U_j},\frac{H_q^i}{\partial U_j}\right]^T$$ i: the no. of loops; j: the no. of dependent variables

$$
\begin{array}{ccccccccc}
\mathbf{u_1} & \mathbf{u_2} & \mathbf{u_3} & \mathbf{u_4} & \mathbf{u_5} & \boldsymbol{\phi_1} & \boldsymbol{\phi_2} & \boldsymbol{\phi_3} & \boldsymbol{\phi_4}
\end{array}
$$

$$
=\begin{bmatrix}
0 & 0.9647 & 0 & 0 & 0 & 18.7181 & 10.0477 & 0 & 0 \\
-1 & 0.2635 & 1 & 0 & 0 & -6.6200 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0.9647 & 0 & 0 & 10.0477 & 4.0600 & 0 \\
0 & 0 & 1 & 0.2635 & 0 & 0 & 0 & -3.9050 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0.9647 & 0 & 10.0477 & 0 & 10.6750 \\
0 & 0 & 1 & 0 & 0.2635 & 0 & 0 & 0 & -28.1250 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1
\end{bmatrix} \qquad (28)
$$

$$[S] = -[B]^{-1}[A]$$

$$= \begin{bmatrix} 1.3098 & 1.0367 & 0.2581 & 0.7419 & -0.0705 & -0.2731 \\ 1.3097 & 0 & 0.3453 & -0.3453 & -0.0943 & 0 \\ 0 & 1.0367 & -0.0872 & 1.0872 & 0.0238 & -0.2731 \\ 0 & -0.2731 & -0.2385 & 0.2385 & 0.0651 & 1.0366 \\ 0 & -0.2731 & 0.0250 & -0.0250 & 1.0298 & 1.0366 \\ 0 & 0 & -0.0384 & 0.0384 & 0.0105 & 0 \\ 0 & 0 & 0.0384 & -0.0384 & -0.0105 & 0 \\ 0 & 0 & -0.0384 & 0.0384 & 0.0105 & 0 \\ 0 & 0 & -0.0384 & 0.0384 & 0.0105 & 0 \end{bmatrix} \quad (29)$$

The variations of the kinematic or assembly variables can then be calculated by applying equation (13) or (14).

Worst case: Statistical model:

$$\begin{Bmatrix} \Delta\mathbf{U}_1 \\ \Delta\mathbf{U}_2 \\ \Delta\mathbf{U}_3 \\ \Delta\mathbf{U}_4 \\ \Delta\mathbf{U}_5 \\ \Delta f_1 \\ \Delta f_2 \\ \Delta f_3 \\ \Delta f_4 \end{Bmatrix} = \begin{Bmatrix} 0.5421 \\ 0.3899 \\ 0.2942 \\ 0.2384 \\ 0.5174 \\ 0.8156° \\ 0.8156° \\ 0.8156° \\ 0.8156° \end{Bmatrix} \qquad \begin{Bmatrix} \Delta\mathbf{U}_1 \\ \Delta\mathbf{U}_2 \\ \Delta\mathbf{U}_3 \\ \Delta\mathbf{U}_4 \\ \Delta\mathbf{U}_5 \\ \Delta f_1 \\ \Delta f_2 \\ \Delta f_3 \\ \Delta f_4 \end{Bmatrix} = \begin{Bmatrix} 0.2998 \\ 0.2725 \\ 0.1844 \\ 0.1411 \\ 0.3836 \\ 0.4784° \\ 0.4784° \\ 0.4784° \\ 0.4784° \end{Bmatrix} \quad (30)$$

In this assembly, dimension $\mathbf{U_1}$ is the one for which a design tolerance was specified, since its value and variation will affect the desired performance of the assembly. If the design limits for $\mathbf{U_1}$ are set to be $T_{ASM} = \pm\ 0.28$ and the estimated variation $\Delta\mathbf{U_1}$ represents 3.0 standard deviations, then the design spec corresponds to Z standard deviations, where:

$$Z = \frac{T_{ASM}}{\sigma_1} = \frac{0.28}{0.2998} * 3.0 = 2.8019 \quad (31)$$

Then, the predicted reject rate on each design limit is estimated from

Reject Rate = $\eta$ = 0.002540 dpu

Assembly Rejects = $2\eta$ * Number of the Assemblies

= 2(0.002540)1000

= 5.08 per 1000 assemblies (32)

## 8. Conclusions

The Direct Linearization Method has been presented as a comprehensive method for 2-D assembly tolerance analysis. It meets many of the requirements stated in the introduction.

1. It provides a statistical method for analyzing assemblies with implicit kinematic assembly constraints--even systems of implicit constraint equations.

2. It is capable of representing the three main sources of variation in mechanical assemblies: dimensional, geometric and kinematic.

3. The method is computationally efficient, making it suitable for design iteration and optimization.

4. It offers a comprehensive system for describing a wide variety of assembly applications using a few basic engineering elements.

5. The generalized approach is suitable for computer automation of many of the tasks of assembly tolerance modeling and analysis.

6. The system has been integrated with commercial CAD systems. Assembly tolerance models can be created graphically, with dimensional and tolerance data extracted directly from the CAD database.

This paper has presented a comprehensive method for assembly tolerance modeling and analysis. It will make possible new CAD tools for engineering designers which integrate manufacturing considerations into the design process. Using this tool, designers will be able to quantitatively predict the effects of variation on performance and producibility. Design and manufacturing personnel can adopt a common engineering model for assemblies as a vehicle for resolving their often competing tolerance requirements. Tolerance analysis can become a common meeting ground where they can work together to systematically pursue cost reduction and quality improvement.

Testing of the DLM assembly tolerance analysis method has shown that it produces accurate evaluations for engineering designs [Gao 1993].

In order to simplify the procedure, only dimensional tolerances and 2-D assemblies were discussed in this paper. With appropriate modifications, the DLM can also be applied to 3-D assemblies, including geometrical feature tolerances and kinematic adjustments. These results will be presented in a future paper.

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