Including Geometric Feature Variations in Tolerance Analysis of Mechanical Assemblies

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Abstract

Geometric feature variations are the result of variations in the shape, orientation or location of part features as defined in ANSI Y14.5M-1982 tolerance standard [ANSI 1982]. When such feature variations occur on the mating surfaces between components of an assembly, they affect the variation of the completed assembly. The geometric feature variations accumulate statistically and propagate kinematically in a similar manner to the dimensional variations of the components in the assembly.

The Direct Linearization Method (DLM) for assembly tolerance analysis provides a method for estimating variations and assembly rejects, caused by the dimensional variations of the components in an assembly. So far, no generalized approach has been developed to include all geometric feature variations in a computer-aided tolerance analysis system.

This paper introduces a new, generalized approach for including all the geometric feature variations in the tolerance analysis of mechanical assemblies. It focuses on how to characterize geometric feature variations in vector-loop-based assembly tolerance models. The characterization will be used to help combine the effects of all variations within an assembly in order to predict assembly rejects using the DLM.

1. Introduction

Tolerance analysis of mechanical assemblies is an essential step in the design and manufacturing of high quality products. The appropriate allocation of tolerances among the separate parts in an assembly can result in lower costs per assembly and higher probability of fit, reducing the number of rejects or the amount of rework required on components. Analyzing the cumulative effects of component tolerances on critical clearances or fits in the assembly is necessary to guarantee that the product will function properly.

Unfortunately, tolerance analysis generally involves complex and tedious calculations which are time-consuming and prone to error. In an attempt to solve this problem, efforts are being made to automate the tolerance analysis and allocation process as much as possible. With the increasing use of CAD/CAM/CAE systems for mechanical design, it is desirable to use a CAD model of the assembly to perform the tolerance analysis at the design stage.

The authors' previous paper [Chase, Gao & Magleby 1994] developed a comprehensive system which is capable of performing assembly tolerance analysis by the Direct Linearization Method (DLM), using a vector-loop-based assembly tolerance model. It is a generalized approach employing vector loops, kinematic joints and component tolerances. However, in the previous paper, only component dimensional tolerances were included in evaluating the assembly variations and predicting assembly rejects. The geometric feature tolerances of components in the assembly were not addressed.

Besides dimensional variations, the fluctuation in manufacturing conditions can also cause geometric feature variations, such as the variation of the form of a feature as compared to perfect form, for example. The geometric feature variations of a part can affect the position and orientation of mating parts, and therefore, have the possibility of greatly affecting the final assembly due to the accumulation of individual geometric feature variations. Since this variation is inevitable in manufacturing, it must be carefully controlled in order to produce assemblies which function properly.

The two-component assembly shown in Figure 1 demonstrates the relationship between dimensional variations in an assembly and the small kinematic adjustments which occur at assembly time. The assembly has three component dimensions that vary, two on the tapered groove and one on the cylinder, as shown. The variations in the three dimensions have an effect on the distance U. U is important to the function of the assembly and will be referred to as an assembly resultant.

The parts are assembled by inserting the cylinder into the groove until it makes contact on the two mating surfaces. For each set of parts, the distance U will adjust to accommodate the current value of dimensions A, R, and θ. The assembly resultant U_1 represents the nominal position of the cylinder, while U_2 represents the position of the cylinder when the variations are present. This adjustability of the assembly describes a kinematic constraint, or a closure constraint on the assembly.

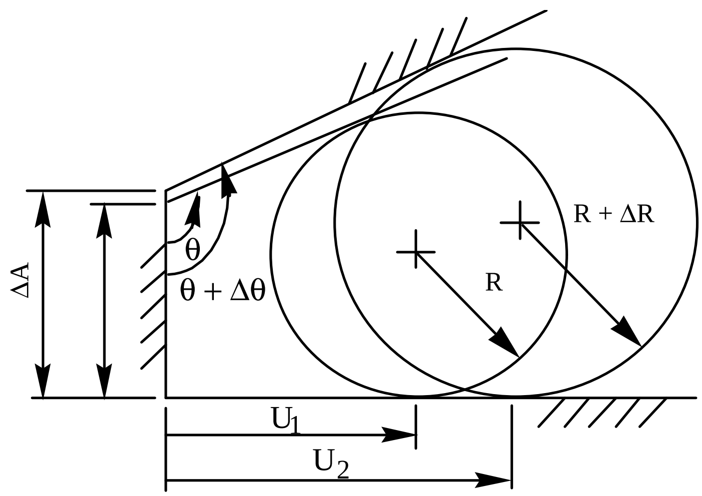

Figure 1. Kinematic adjustment due to component dimensional variations

Figure 2 illustrates the same assembly with exaggerated geometric feature variations. For production parts, the contact surfaces are not really flat and the cylinder is not perfectly round. The pattern of surface waviness will differ from one part to the next. In this

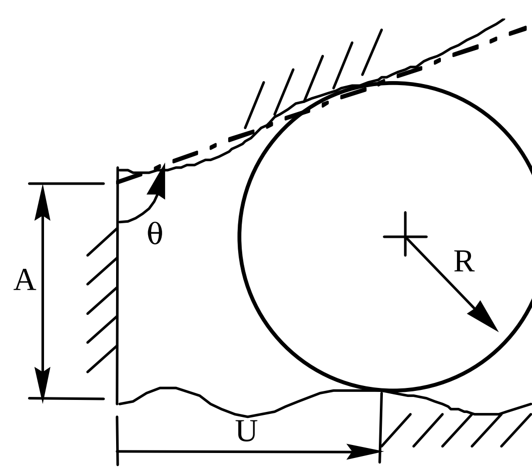

Figure 2. Adjustment due to geometric shape variations

assembly, the cylinder makes contact on a peak of the lower contact surface, while the next assembly may make contact in a valley. Similarly, the lower surface is in contact with a lobe of the cylinder, while the next assembly may make contact between lobes.

Local surface variations such as these can propagate through an assembly and accumulate just as dimensional variations. Thus, in a complete assembly model all three sources of variation, that is, dimensional and geometric feature variations and kinematic adjustments, must be accounted for to assure realistic and accurate results.

The objective of the research described in this paper is to include geometric feature variations of components in a vector-loop-based assembly tolerance model. The general approach includes the characterization of geometric feature variation in 2-D and 3-D assemblies, and using the DLM to calculate the variation of assembly dimensions or kinematic variables caused by geometric feature tolerances.

2. Definitions

The geometric feature tolerances defined by ANSI Y14.5M-1982 fall into five main groups, according to Foster [1992]:

1. ***FORM*** A form tolerance states how far an actual surface or feature is permitted to vary from the desired form implied by the drawing. It includes flatness, straightness, circularity and cylindricity.

2. ***PROFILE*** A profile tolerance states how far an actual surface or feature is permitted to vary from the desired form on the drawing and/or vary relative to a datum or datums. Profile of a line and profile of a surface are the only two types of profile tolerance.

3. ***ORIENTATION*** An orientation tolerance states how far an actual surface or feature is permitted to vary relative to a datum or datums. It consists of perpendicularity, angularity and parallelism.

4. ***LOCATION*** A location tolerance states how far an actual size feature is permitted to vary from the perfect location implied by the drawing as related to a datum, or datums, or other features. This category includes position and concentricity.

5. ***RUNOUT*** A runout tolerance states how far an actual surface or feature is permitted to vary from the desired form implied by the drawing during full (360°) rotation of the part on a datum axis. A runout can be either a circular runout or a total runout.

Figure 3 illustrates the symbols which represent the geometric feature controls defined in the ANSI standard.

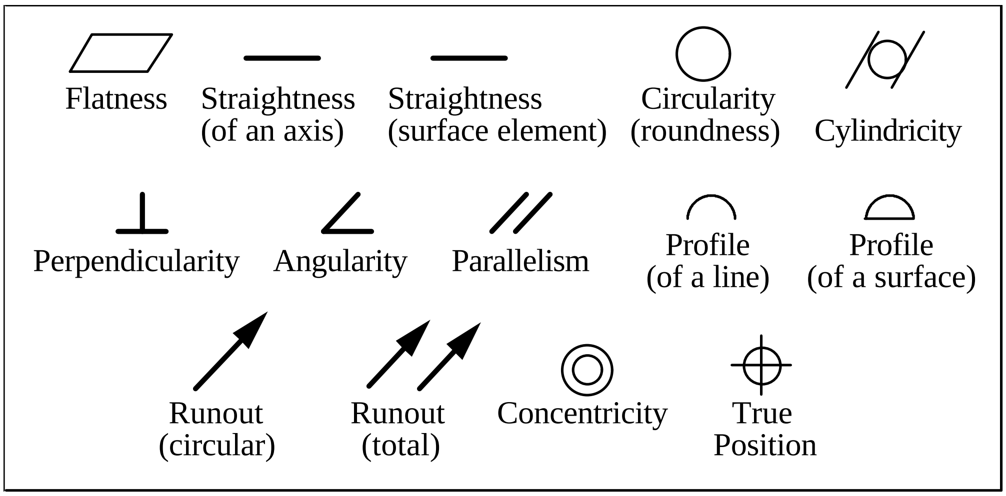

Figure 3. Symbols for geometric feature controls [ANSI Y 14.5M-1982].

Geometric feature controls allow the designers to specify limits on the form or orientation of a feature on a part, which are not available through the use of size tolerances alone. Foster [1986] listed the conditions under which it is appropriate to use geometric feature tolerancing:

1. Whenever part features are critical to function or interchangeability.
2. Whenever functional gaging techniques are desirable.
3. Whenever datum reference frames are desirable in order to ensure consistency between manufacturing and gaging operations.
4. Whenever computerization techniques in design and manufacturing are desirable.
5. Whenever the standard interpretation or tolerance is not already implied.

3. Research Review

There are two basic issues which must be resolved when including geometric feature variations in assembly tolerance analysis. First, how to represent or characterize the geometric feature variations in an assembly tolerance model, and second, how to evaluate the effects of the geometric feature variations when estimating assembly variation and predicting the percentage of assemblies which will fall outside of the design specifications.

3.1 Tolerance Representation

Schemes for tolerance representations or characterizations in an assembly have been developed with the increasing use of solid modeling tools in product design. These schemes can be generally classified into three groups:

1. Set theoretic model
2. Offset zones
3. Parametric zones

The set theoretic model of tolerances describes a variational class of objects (or parts) which is defined by the tolerances applied to the nominal object. This variational class is modeled as a set of points in 3-D space, which contains the nominal object but does not force any part of the object's real boundary to be in an exact position [Shah & Miller 1990]. The set theoretic model has not been implemented because it is difficult to mathematically describe objects in terms of its theory [Robison 1989].

Offset zones are created by offsetting the nominal boundary of a part by an amount equal to the tolerance on either side of the nominal [Requicha 1983]. Offsets are obtained for the maximum material condition (MMC) and for the least material condition (LMC). The difference between these two zones comprises the tolerance zone, an envelope within which the boundary of the part must lie [Shah & Miller 1990]. This method seems to lend itself to the use of "go-nogo" gages to check the tolerance condition of a part. A disadvantage of this method is that it assumes that all surfaces remain in the same orientation as the nominal surface. It has not be used to model variations in the orientation of a surface, such as angularity [Robison 1989].

A parametric zone or space is composed of a set of parameters or dimensions and constraints which describe the nominal shape of the geometry [Hillyard & Braid 1978, Martino & Gabriele 1989]. Tolerances are treated as small variations in these parameters. This type of tolerance model is closely related to the variational geometry approach for CAD modelers [Shah & Miller 1990, Guilford & Turner 1993]. The advantage of this model is that it uses the constraints and parameters of the geometry to create a set of equations which may be solved to determine any unknown dimensions or variations [Gupta & Turner 1993].

A combination of parametric zone and offset zone for representing tolerances in an assembly has been recently proposed by Gilbert [1992]. They use the 4x4 homogeneous transformation matrix to contain the nominal relations between parts and variations allowed by the tolerances in an assembly tolerance model. Most geometric feature variations, except for form tolerances, can be represented by this method.

The assembly tolerance analysis model adopted by this paper is a parametric zone type, proposed by Robison [1989]. It is composed of a vector-based method for modeling 3-D mechanical assemblies, which utilizes vectors to represent dimensions between critical part features and includes a set of kinematic joint types to represent mating conditions between parts at the contact locations [Chase, Gao & Magleby 1994]. This method also includes guidelines for identifying a valid set of vector loops to ensure that the tolerance model is complete. This method lays the vector-loop-based assembly tolerance model over the solid model, and can be connected with a tolerance analysis package to solve for the variations on the desired dimensions or clearances. This model is also capable of including component geometric feature tolerances.

3.2 Tolerance Analysis

After specifying the variation of the individual components in an assembly, the propagation of the variations in the assembly must be determined. This is frequently referred as “tolerance analysis.”

The results of a tolerance analysis are estimates of the mean, variance and other statistical parameters describing the variation of critical assembly features. If engineering design limits have been specified for a feature, quantitative estimates of the percent rejects may be made and compared to desired quality levels.

Five methods have been employed for 2-D and 3-D tolerance analysis: 1) Linearized or Root Sum Square, 2) Method of System Moments, 3) Hasofer-Lind Reliability Index, 4) Direct Integration and 5) Monte Carlo Simulation. These have been discussed in previous papers [Chase & Parkinson 1991, Chase, Gao & Magleby 1994]. All five methods have been primarily limited to problems for which the assembly resultant of interest may be expressed <u>explicitly</u> as a function of the dimensions of the components in an assembly. Establishing such an explicit function for a real assembly is very difficult or impossible for the designer. In addition, the emphasis of these methods is the propagation of dimensional variations rather than geometric feature variations in an assembly.

Gilbert [1992] and Whitney, et. al [1994] proposed a system to estimate the propagation effect of the dimensional and geometric feature tolerances of components in an open loop assembly using the linearized model of the 4x4 transformation matrix constraint developed by Veitschegger and Wu [1986]. He studied most geometric feature tolerances, except for form variations, and characterized the variation types of these geometric feature variations for some mating conditions between the parts.

Lin and Chen [1994] developed a linearized scheme for tolerance analysis of closed loop mechanisms in 3-D space using the 4x4 transformation matrix in the Denavit-Hartenberg (D-H) symbolic notation common to robotics. This method is significant since it has the capability of assessing the influence of each error component on a machine's accuracy, and it is also capable of including geometric feature variations. It is particularly useful to allocate a machine's tolerances at its design stage and to characterize the performance of an existing machine. Thus far, the method has not been applied to static assemblies. It has only been applied to robotic mechanisms with revolute and prismatic joints (see joint classifications in a later section).

The authors' previous paper [Chase, Gao & Magleby 1994] developed the Direct Linearization Method for treating implicit assembly functions, such as kinematic constraints, for assembly tolerance analysis in 2-D space. It is a generalized approach, employing common engineering concepts of vector loops, kinematic joints and geometric feature tolerances.

The DLM reduces the implicit equations describing the kinematic constraints to a set of linear equations for small changes about the nominal, which may be solved directly for the assembly variances. Assembly geometry is reduced dramatically, so it is very efficient computationally. It is suited for integration with commercial CAD systems, by which the required geometry may be extracted directly from a solid model. This paper extends the DLM method to include all geometric feature variations in an assembly.

A comparison of the accuracy of linearized tolerance analysis with Monte Carlo simulation was performed earlier [Gao, et. al 1995, Gao 1993]. Several assembly problems, covering a variety of critical assembly features, were analyzed by both methods and the results were compared. The mean, standard deviation and percent rejects were computed for each case. The results of the study demonstrated that the linearized method has an accuracy equal to Monte Carlo. While the linearized method requires a sample size of one, it was shown to be equivalent to a Monte Carlo analysis with a sample size of 10,000.

4. The Direct Linearization Method

The assembly constraint equations should be able to describe the small kinematic adjustments of the assembly resultants or assembly variables occurring at assembly time, due to manufacturing variations on the component dimensions. In an assembly, the kinematic constraint may appear as a closed loop, that is, the starting and ending point of the vector loop for the assembly is the same point, or as an open loop, that is, the assembly vector loop ends with a gap or clearance. This constraint with vector loop-based assembly models may be expressed as a concatenation of homogeneous transformation matrices:

$$[R_1][T_1][R_2][T_2]...[R_i][T_i]...[R_n][T_n][R_f] = [H] \qquad (1)$$

where $[R_i]$ is the rotational transformation matrix or product of rotational transformation matrices and $[T_i]$ is the translational matrix at node i, $[R_f]$ is the final closure rotation, and [H] is the resultant matrix. If the assembly is described by closed loop constraints, [H] = [I], the identity matrix, otherwise [H] represents the final gap or clearance and its orientation.

For 2-D assemblies:

$$R = \begin{bmatrix} \cos f_z & -\sin f_z & 0 \\ \sin f_z & \cos f_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & 0 & T_x \\ 0 & 1 & T_y \\ 0 & 0 & 1 \end{bmatrix} \qquad (2)$$

For 3-D assemblies:

$$R_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos f_x & -\sin f_x & 0 \\ 0 & \sin f_x & \cos f_x & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_y = \begin{bmatrix} \cos f_y & 0 & \sin f_y & 0 \\ 0 & 1 & 0 & 0 \\ -\sin f_y & 0 & \cos f_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (3)$$

$$R_Z = \begin{bmatrix} \cos f_z & -\sin f_z & 0 & 0 \\ \sin f_z & \cos f_z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where ϕ_X, ϕ_Y and ϕ_Z are relative rotations about their corresponding axes and T_X, T_Y and T_Z are components of the translation vector from one node in the loop to the next.

Equations (1) are nonlinear and require nonlinear solver to find solutions. However, for small variation about the nominal of each component in the assembly, the equations can be approximated by the linearized Taylor expansion of equation (1). Therefore, only the derivatives are needed.

Robison [1989] used a perturbation method for evaluating the derivatives of the assembly constraint equation (1). For a 3-D case, if the translation or rotation at joint i is the variable with respect to which the derivatives are desired, a small length perturbation δL or angle perturbation $\delta\phi$ is added to the original value, then the matrix multiplication of equation (1) is performed:

for translation

$$[R_1][T_1]...[R_i][T_i(L+\delta L)]...[R_n][T_n]\{0\ 0\ 0\ 1\}^t = \{\Delta X\ \Delta Y\ \Delta Z\ 1\}^t \tag{4}$$

for rotation

$$[R_1][T_1]...[R_i(\phi + \delta\phi)][T_i]...[R_n][T_n]\{0\ 0\ 0\ 1\}^t = \{\Delta X\ \Delta Y\ \Delta Z\ 1\}^t \tag{5}$$

the derivatives can then be approximated numerically by:

Translational variable	Rotational variable
$\frac{\partial H_x}{\partial L} \cong \frac{\Delta X}{dL}$	$\frac{\partial H_x}{\partial f} \cong \frac{\Delta X}{df}$
$\frac{\partial H_y}{\partial L} \cong \frac{\Delta Y}{dL}$	$\frac{\partial H_Y}{\partial f} \cong \frac{\Delta X}{df}$

$$\frac{\partial H_z}{\partial L} \cong \frac{\Delta Z}{dL} \qquad \frac{\partial H_Z}{\partial f} \cong \frac{\Delta X}{df} \tag{6}$$

$$\frac{\partial H_{qx}}{\partial L} = 0 \qquad \frac{\partial H_{qx}}{\partial f} = COS(a)$$

$$\frac{\partial H_{qY}}{\partial L} = 0 \qquad \frac{\partial H_{qY}}{\partial f} = COS(b)$$

$$\frac{\partial H_{qZ}}{\partial L} = 0 \qquad \frac{\partial H_{qZ}}{\partial f} = COS(g)$$

where H_X, H_Y and H_Z are the translational constraint, and $H_{\theta X}$, $H_{\theta Y}$ and $H_{\theta Z}$ are the rotational constraint in x, y and z directions respectively, and α, β and γ are the global direction cosine angles of the local axis around which the rotation is made [Gao 1993].

The DLM uses the linearized Taylor expansion of equation (1) to solve for the assembly variations. The linearized constraint equations for the closed loops in an assembly may be expressed in matrix form:

$$\{\Delta H\} = [A]\{\Delta X\} + [B]\{\Delta U\} = \{0\} \tag{7}$$

where {ΔH}: the variations of the clearance,
{ΔX}: the variations of the manufactured variables,
{ΔU}: the variations of the assembly variables,
[A]: the partial derivatives with respect to the manufactured variables,
[B]: the partial derivatives with respect to the assembly variables,

If the [B] is a full-ranked matrix, the variations of the assembly or kinematic variables can be obtained by solving equation (7).

$$\{\Delta U\} = -[B]^{-1}[A]\{\Delta X\} \tag{8}$$

If the [B] matrix is singular, the least square fit solution is

$$\{\Delta U\} = -([B]^T[B])^{-1}[B]^T[A]\{\Delta X\} \tag{9}$$

If the assembly has open loop constraints, the first order Taylor series expansion of the assembly constraints is:

$$\{\Delta\Phi_p\} = [C]\{\Delta X\} + [D]\{\Delta U\} \quad (10)$$

where $\{\Delta\phi_p\}$: the variations of the open loop assembly variables,

[C]: the partial derivatives with respect to the manufactured variables in the open loop,

[D]: the partial derivatives with respect to the assembly variables in the open loop,

If one substitutes $\{\Delta U\}$from equation (8) or (9) into equation (10), then the variations of the open loop variables can be expressed as:

$$\{\Delta\Phi_p\} = ([C] - [D][B]^{-1}[A])\{\Delta X\} \quad (11)$$

If [B] is singular,

$$\{\Delta\Phi_p\} = \left([C]-[D]([B]^T[B])^{-1}[B]^T[A]\right)\{\Delta X\} \quad (12)$$

Equations (8) through (12), above, may be modified to obtain estimates of the tolerance accumulation in the assembly by worst case or statistical methods. Expressions for each method are presented in a later section. Both will be augmented to include geometric as well as dimensional variations.

5. Characterizing Geometric Feature Variations

The geometric feature variations defined in the ANSI standard must be modeled so that their effects will be reflected in the tolerance model of the assembly. By analyzing the assembly constraint equations, the effects of the geometric feature variations on the assembly or kinematic variables can then be estimated (see section 5). In the vector-loop-based assembly tolerance model, this is done by modeling the geometric feature variations with zero length vectors having specified variations or tolerances, placed at the contact point between mating surfaces. These zero length vectors are considered as independent variation sources to the dimensional variations in the assembly. The direction in which they introduce variation into an assembly depends on the type of contact which exists between the surfaces. For this reason, geometric feature tolerances of components in an assembly are related to the joint types through which the geometric feature variations are propagated.

5.1. Geometric Feature Tolerance Modeling in 2D

The kinematic joint type and geometric feature tolerances on the parts in contact are the key elements in analyzing the effect of the geometric feature tolerances on assembly variations. The commonly used kinematic joint types in 2-D space are modeled in Figure 4 [Chase, Gao & Magleby 1994, Chun 1988]. The effect of the geometric feature tolerances associated with each of the joints may result in translational variation or rotational variation. This translational or rotational variation is usually smaller than the size tolerances on the same parts.

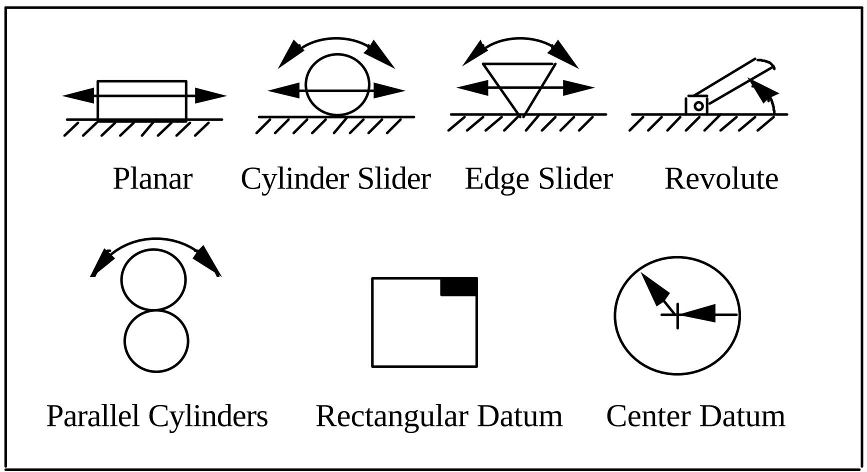

Figure 4. Kinematic joint and feature datum types in 2-D space

Figure 5 illustrates how a flatness tolerance zone can affect two mating parts differently when viewed in 2-D. The cylinder on the left illustrates a translational variation, while the block on the right exhibits the rotational variation, due to the same geometric feature variation. The translation of the planar joint is reflected in the dimension variations, except for runout and concentricity. So, the nature of the contact between mating surfaces determines how feature variations propagate through an assembly.

The rotation variation for the block on a plane surface in Figure 5 is related to the flatness tolerance zone and the contact length of the block, in this case, the horizontal dimension of the block. This contact length is called <u>characteristic length</u>.

$$\Delta\beta = \mathrm{Tan}^{-1}\left(\frac{\text{Flatness Tolerance Zone}}{\text{Characteristic Length}}\right) \qquad (13)$$

where $\pm\Delta\beta$ is the rotational variation caused by the flatness in a planar joint. The translational variation is $\pm\alpha/2$ where α is the width of the tolerance band.

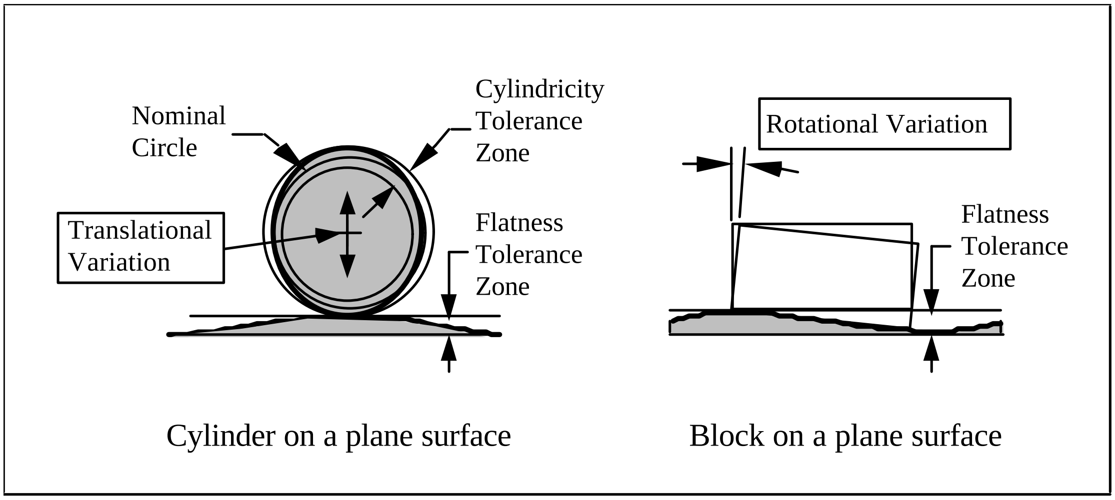

Figure 5. 2-D effects of geometric feature tolerance

If the translational variation caused by the geometric feature tolerance is represented by **T** and rotational variation by **R**, all the possible combinations of the geometric feature tolerances with the kinematic joint types can be summarized in Table 1. The empty cells in the table mean that the corresponding geometric feature tolerance and kinematic joint combination does not apply.

Table 1 Rotational and translational variations associated with corresponding geometric feature tolerance-kinematic joint combinations in 2-D

Joints \ Geom Tol	⏥	—	○	⌭	⌓	//	⊥	∠	⌰	◎	⌖
Planar	R	R			R	R	R	R	R T	R T	
Cyl Slider	T	T	T	T	T	T	T	T	T		
Edge Slider	T	T	T	T	T	T	T	T	T	T	T
Revolute										T	T
Par Cylind		T	T	T	T				T		

Note: For a rotational variation, a characteristic length should be provided for each feature tolerance associated with it.

A debatable point is whether or not rotational variations should also include a translation normal to the surface. That is, does a block on a plane exhibit both rotation and translation due to surface variations? The answer depends on how the surface is manufactured and how it will be inspected. If there is a size dimension normal to the plane, say, describing plate thickness, the size variation will probably include translation variations. If the inspection for size involves area contact between the instrument and the surface, the highest waviness peak would determine the size. Thus, waviness would be included in the size.

The exceptions to this interpretation are runout and concentricity. Consider inspecting a platter of a computer disk drive. The waviness period associated with runout is generally much larger than the contact area of either the thickness measuring instrument or the read-write head. Thus, the waviness would introduce a translational variation independent of size variation.

On the other hand, rotational variation is also dependent on the period of the waviness. If the surface waviness for the process has a much longer or much shorter period than the length of contact between mating surfaces (characteristic length), little or no rotation will occur. Such differences can be taken into account by adjusting the width of tolerance band or the characteristic length to produce variations in the model which truly represent the $\pm 3\sigma$ variations exhibited by the assembled parts. Extensive research is needed in this area to properly characterize geometric variational effects between mating parts.

5.2. Geometric Feature Tolerance Modeling in 3-D

The effects of a geometric feature tolerance on a kinematic joint in 3-D are more complicated than in 2-D, because a three dimensional local joint coordinate reference frame is required to describe the position and orientation of the joint. The most commonly used kinematic joints in 3-D are illustrated in Figure 6 [Robison 1989, Goodrich 1991, Ward 1992].

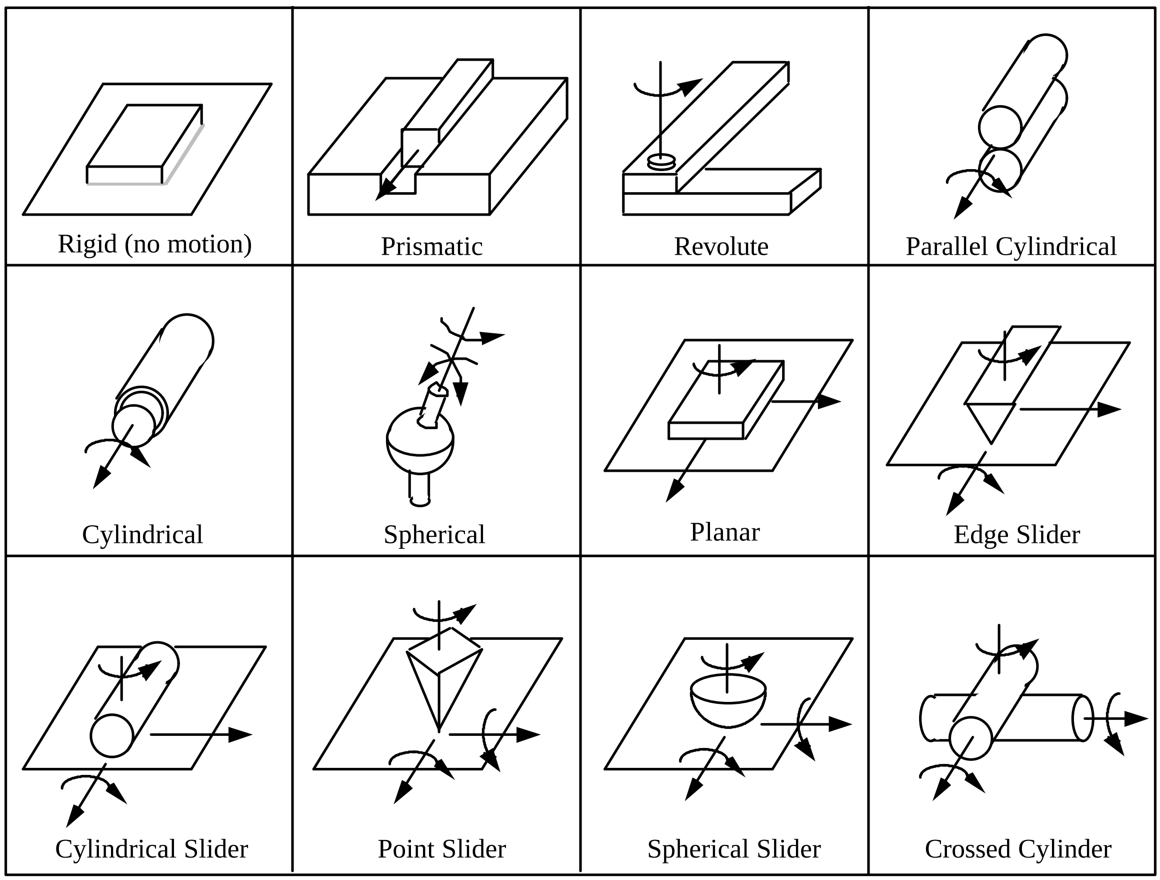

Figure 6. Kinematic joint types in 3-D

Each geometric feature tolerance may produce variations, either translation, rotation or both, at the joint in different directions. If the geometric feature tolerance of a cylinder slider joint is extended to 3-D, the position and orientation of the cylinder will be affected simultaneously. Viewing along the axis of the cylinder (local z axis), the cylinder may rest on a peak or in a valley of the plane and that will result in a translational variation normal to the plane. While, looking perpendicular to the cylinder (local x axis), one end of the cylinder may lie in a valley and the other end on a peak of a surface and this will cause a rotational variation about the local x axis (see Figure 7).

For example, in Figure 7, if the translational variation in the local joint coordinate reference frame is represented by T_y and the rotational variation by R_x, the combination of the cylindrical slider and plane geometric feature tolerance flatness will produce the combined variations (T_yR_x).

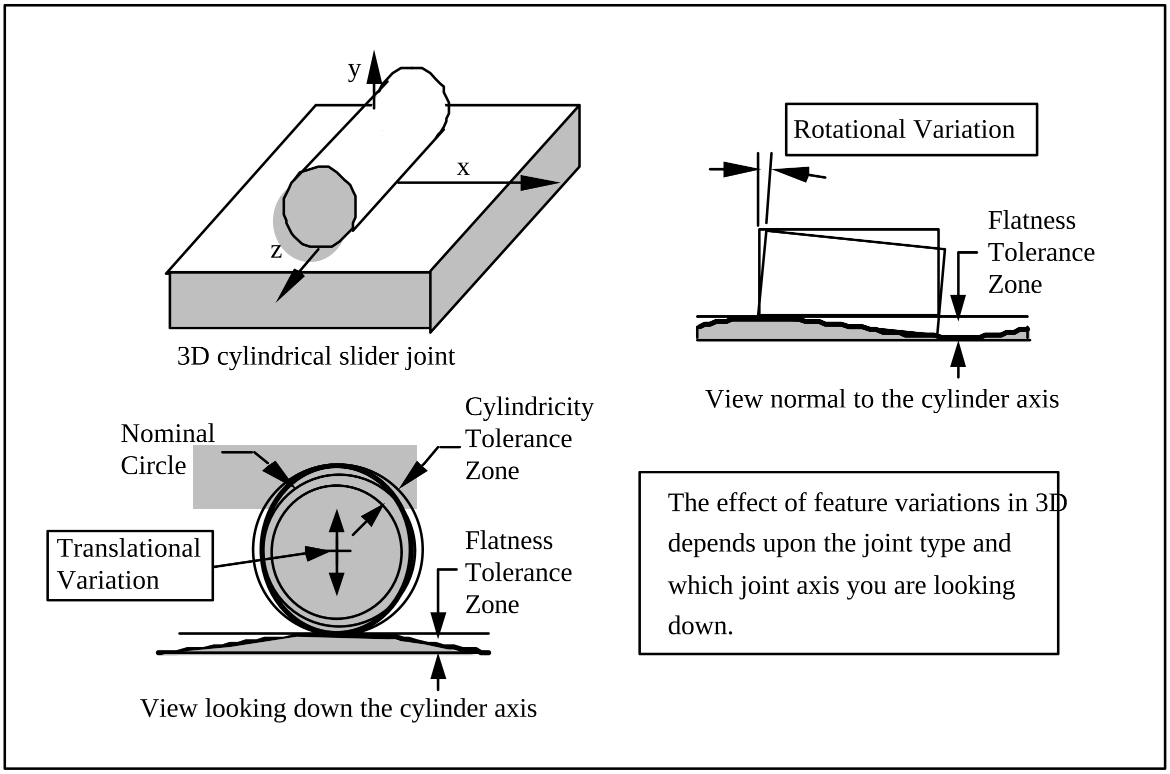

Figure 7. 3-D effects of geometric feature tolerance

Goodrich [1991], Ward [1992] and Gao [1993] studied all the possible combinations of the kinematic joints and geometric feature tolerances in 3-D space (see Table 2). These variation combinations can then be used to estimate the assembly or kinematic variations caused by the geometric feature tolerances.

It is important to point out that the x, y and z axis for each of these joints is a local axis specific to each joint (see Figure 7 and 8). From Figure 8, it can be seen that the degrees of freedom for kinematic motion (K) and the degrees of freedom for feature variations (F) are mutually exclusive. Everywhere K is removed by constraints, there is possibility for F. This fact is true for all of the kinematic joints listed in Figure 6.

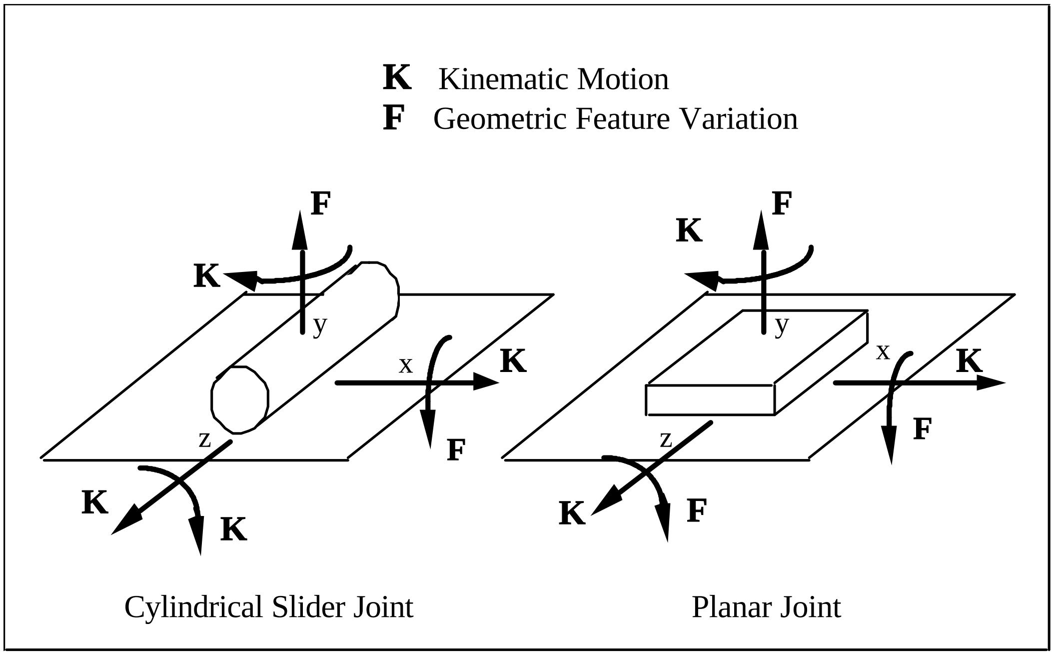

Figure 8. Degrees of freedom for kinematic motions and geometric feature variations.

Table 2 Rotational and translational variations associated with corresponding geometric feature tolerance-kinematic joint combinations in 3-D

Joints \ Geom Tol		▱	—	○	⌭	⌓	∥	⊥	∠	⌰	◎	⌖
Planar		Rx Rz	Rx Rz			Rx Rz	Rx Rz	Rx Rz	Rx Rz	RxRzTy		
Revolute			Rx Rz	Rx Rz	Rx Rz	Rx Rz	Rx Rz	Rx Rz	Rx Rz	Rx Rz	Tx Tz	Tx Tz
Cylindrical			Rx Rz	Rx Rz	Rx Rz	Rx Rz	Rx Rz	Rx Rz	Rx Rz	Rx Rz	Tx Tz	Tx Tz
Prismatic		RxRyRz	RxRyRz			RxRyRz	RxRyRz	RxRyRz	RxRyRz			
Spherical				TxTyTz		TxTyTz				TxTyTz	TxTyTz	TxTyTz
Crscyl			Ty	Ty	Ty	Ty				Ty		Ty
Parcyl			Ty Rx	Ty Rx	Ty Rx	Ty Rx	Rx	Rx	Rx	Ty Rx		Ty
Edgsli	E		Ty Rx			Ty Rx	Ty Rx	Ty Rx	Ty Rx			Ty
	P	Ty Rx	Ty Rx			Ty Rx	Ty Rx	Ty Rx	Ty Rx	Ty Rx		
Cylsli	C		Ty Rx	Ty Rx	Ty Rx	Ty Rx	Ty Rx	Ty Rx	Ty Rx	Ty Rx		Ty
	P	Ty Rx	Ty Rx			Ty Rx	Ty Rx	Ty Rx	Ty Rx	Ty Rx		
Pntsli	Pt											Ty
	P	Ty	Ty			Ty	Ty	Ty	Ty	Ty		
Sphsli	S			Ty		Ty				Ty		Ty
	P	Ty	Ty			Ty	Ty	Ty	Ty	Ty		

Note: Crscyl = Crossed cylinders
Parsyl = Parallel cylinders
Edgsli = Edge slider (E for edge and P for plane)
Cylsli = Cylindrical slider (C for cylinder and P for plane)
Pntsli = Point slider (Pt for point and P for plane)
Sphsli = Spherical slider (S for sphere and P for plane)

A rigid joint is a joint with all kinematic degrees of freedom removed. Therefore, all axes are possible sources for geometric feature variations. Applicable degrees of freedom depend on the contact existing between mating surfaces. For example, if two plates are welded or fastened, they still have position and orientation variations.

5.3 Application of the Envelope Rule

The foregoing discussion defined a comprehensive system for modeling geometric variations in assemblies. A complete discussion must address the interaction of geometric and size variations.

The geometric tolerance standard ANSI Y14.5M-82 defines Rule #1 as: "the surface of a feature shall not extend beyond a boundary (envelope) of perfect form at MMC." This rule is also called the Envelope Rule. It states that all points on the surface of a feature must lie within the limits defined by its MMC envelope (Maximum Material Condition). This means that any surface variations due to flatness, parallelism, etc. must reduce to zero as the size approaches MMC.

The envelope rule applies to worst case conditions. It was established to permit part inspection with "go-nogo" gages. Go-nogo gages are manufactured to the MMC and LMC values for the dimension being checked and are used for 100% inspection of parts.

The envelope rule should not be applied to statistical variations. Statistical analysis of variation is based upon a distribution centered about a nominal or ideal value. Independent random variations of the component dimensions are the result of process variations and add as root-sum-squares, with the resulting distribution extending theoretically to infinity. Statistical analysis of variation always predicts a certain percentage of parts will fall outside of the design limits. There is no way to apply the envelope rule to a statistical model. Therefore, this paper proposes that the envelope rule only be used for worst case analysis and not be applied to statistical analysis of variation.

The application of the envelope rule to assemblies needs to be generalized, since the envelope rule applies to the individual component dimensions used in the tolerance analysis of an assembly. Procedures for determining when to reduce specific geometric variations in order to avoid violating the MMC envelope for components must be defined for various assembly conditions. The envelope rule does not apply to some geometric variations, such as runout. These cases must be identified and accounted for. The focus of this paper, however, will be on statistical analysis of assemblies. The generalization to worst case analysis is left for a future study.

6. Derivative Evaluation for Geometric Feature Variables

In order to calculate the derivatives for the linearized assembly equations, small variations corresponding to the geometric variations must be introduced at appropriate points in the loop. The effect on loop closure may be calculated and used to calculate the sensitivities, as was done for length and angle sensitivities.

Although geometric variations may affect an entire surface, they can only introduce variation into an assembly at the contact surfaces between mating parts. Geometric disturbances propagate through the joints along the constrained directions, that is, in the direction of the geometric degrees of freedom (F axes in figure 8).

In the matrix loop equation, as the transformation matrices traverse the loop, they transform a local coordinate system from node to node. At each joint, the rotating coordinate system must be oriented to coincide with the local joint reference frame. Appropriate geometric feature variations may be added at the joint as permitted (see Table 1 and Table 2). Each geometric feature variation, is treated as a zero length vector or rotation with a plus or minus variation in the permitted direction.

If the variation caused by the geometric feature tolerance-kinematic joint combination is a rotation, the geometric feature variable can be represented in the assembly kinematic constraints by a rotational matrix or a combination of matrices, as defined in equations (2) and (3). Rotation angles ϕ_x, ϕ_y and ϕ_z are now geometric feature variables with nominal value zero and variation equal to the corresponding rotation tolerance about the given local joint axis.

If the variation caused by the geometric feature tolerance-kinematic joint combination is a translation, a translation matrix, as defined in equations (2) and (3), may be inserted at the appropriate node in the assembly kinematic constraint equation. The translation components T_x, T_y and T_z are the geometric variation components along the local joint axes. They have zero nominal length, but can still introduce a variation along the corresponding joint axes.

The combination of translational and rotational variations caused by the geometric feature tolerance can be handled by combining the above operators together as directed in Table 1 and Table 2 and inserting them into the constraint equations. For the general case, if a geometric feature tolerance is to be added at joint i, the assembly constraint equation (1) becomes:

$$[R_1][T_1]...[R_o][R_{ix}][R_{iy}][R_{iz}][T_i][R_{-o}]...[R_{n+1}][T_{n+1}][R_f] = [I] \quad (14)$$

where $[R_o]$ represents the rotation to orient the rotating coordinate system to coincide with the local joint reference frame and $[R_{-o}]$ is the reverse operation, that is, returning to the orientation just before the geometric feature tolerance was added. Figure 9 illustrates such transformations for the flatness-edge slider combination in 2-D. This process can be repeated throughout the vector loop of the assembly.

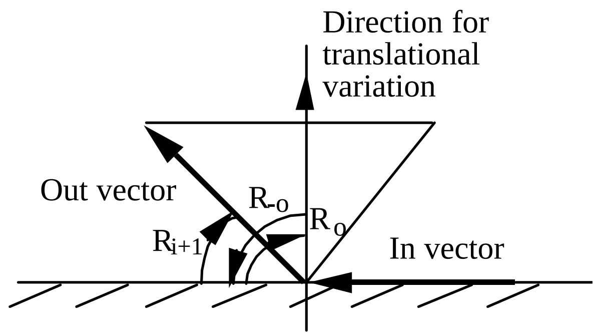

Figure 9. Adding flatness variation to 2-D edge slider joint

After the vector loop is traversed by the above process, the kinematic constraint equation will have a form similar to equation (1), with a few more matrix operations. The numerical derivative evaluation procedure discussed for length and angle variables is also applicable, that is,

for translation

$$[R_1][T_1]...[R_i][T_i(0+\delta L)]...[R_n][T_n]\{0\ 0\ 0\ 1\}^t = \{\Delta X\ \Delta Y\ \Delta Z\ 1\}^t \quad (15)$$

for rotation

$$[R_1][T_1]...[R_i(0 + \delta\phi)][T_i]...[R_n][T_n]\{0\ 0\ 0\ 1\}^t = \{\Delta X\ \Delta Y\ \Delta Z\ 1\}^t \quad (16)$$

where δL is a small change in length in the translation direction and $\delta\phi$ is a small change in the rotation direction. The derivatives can then be approximated numerically by equations (6), as before.

7. Including Geometric Feature Variations in the DLM

The above section described the scheme for characterizing or representing geometric feature variation in 2-D and 3-D assemblies. This enables us to include the effect of geometric feature variation in evaluating the assembly variations.

The DLM uses the linearized Taylor expansion of equation (1) to solve for the assembly variations. If the geometric feature variations of the components in the assembly are included, the linearized constraint equations for an assembly may be modified to include geometric variations:

Closed loops:

$$\{\Delta H\} = [A]\{\Delta X\} + [B]\{\Delta U\} + [F]\{\Delta\alpha\} = \{0\} \quad (17)$$

Open loops:

$$\{\Delta\Phi_p\} = [C]\{\Delta X\} + [D]\{\Delta U\} + [G]\{\Delta\alpha\} \quad (18)$$

where $\{\Delta\alpha\}$: the variations of the geometric feature variables,
[F], [G]: the partial derivatives with respect to the geometric feature variables,
$\{\Delta H\}$, $\{\Delta\Phi_p\}$, $\{\Delta X\}$, $\{\Delta U\}$, [A], [B], [C] and [D] are defined in equations (7) and (10).

Solving equation (17) for $\{\Delta U\}$, the closed loop assembly variations:

$$\{\Delta U\} = -[B]^{-1}[A]\{\Delta X\} - [B]^{-1}[F]\{\Delta\alpha\} \quad (19)$$

If one substitutes $\{\Delta U\}$from equation (19) into equation (18), then the variations of the open loop variables can be expressed as:

$$\{\Delta\Phi_p\} = ([C] - [D][B]^{-1}[A])\{\Delta X\} + ([G] - [D][B]^{-1}[F])\{\Delta\alpha\} \quad (20)$$

8. Estimation of Kinematic Variations and Assembly Rejects

The accumulation of variations in the assembly can then be estimated by applying a worst case or statistical model [Goodrich 1991, Robison 1989] to equations (19) and (20).

Worst case:

$$\Delta U_i = \sum_{j=1}^{n} \left|S_{ij}^{d}\right| Tol_{ij}^{d} + \sum_{j=1}^{m} \left|S_{ij}^{\alpha}\right| Tol_{ij}^{\alpha} \qquad T_{ASM} \tag{21}$$

Statistical model:

$$\Delta U_i = \sqrt{\sum_{j=1}^{n} \left(S_{ij}^{d} Tol_{ij}^{d}\right)^2 + \sum_{j=1}^{m} \left(S_{ij}^{\alpha}\, Tol_{ij}^{\alpha}\right)^2} \qquad T_{ASM} \tag{22}$$

Here S^d is the tolerance sensitivity matrix for the dimensional variables,
S^α is the tolerance sensitivity matrix for the geometric feature variables,
Tol^d is the tolerance vector for the dimensional variables,
Tol^α is the tolerance vector for the geometric feature variable,
n is the number of dimensional variables,
m is the number of geometric feature variables,
T_{ASM} is the design limit for assembly variation ΔU_i.

In equation (22), ΔU_i^2 is an estimate of the statistical variance of assembly variable U_i. If all the component tolerances Tol^d and Tol^α represent $\pm 3\sigma$ variations, ΔU_i will represent the 3σ variation of U_i.

The tolerance sensitivity matrices can be calculated by:

For a well-determined system:

$$S^d = -[B]^{-1}[A] \tag{23}$$

$$S^\alpha = -[B]^{-1}[F] \tag{24}$$

For an over-determined system:

$$S^d = -([B]^T[B])^{-1}[B]^T[A] \tag{25}$$

$$S^\alpha = -([B]^T[B])^{-1}[B]^T[F] \tag{26}$$

Equation (21) or (22) can also be used to estimate the variations of the open loop variables. Here, the sensitivity matrices are:

For a well-determined system:

$$S^d = [C] - [D][B]^{-1}[A] \tag{27}$$

$$S^{\alpha} = [G] - [D][B]^{-1}[F] \quad (28)$$

For an over-determined system:

$$S^{d} = [C] - [D]([B]^{T}[B])^{-1}[B]^{T}[A] \quad (29)$$

$$S^{\alpha} = [G] - [D]([B]^{T}[B])^{-1}[B]^{T}[F] \quad (30)$$

The estimation of the assembly rejects is based on the assumption that the distribution of the assembly variable(s) is Normal, which is a reasonable assumption for most assemblies. The estimate of a kinematic or assembly variation is treated as three standard deviations, and this deviation together with the mean value of the kinematic or assembly variable can be used to calculate by integration or table the assembly rejects for a given assembly batch if the limits for the assembly variable are specified.

9. Example

As an example to demonstrate how to include geometric feature tolerances in the variation estimation of kinematic or assembly variables and prediction of assembly rejects, the tape hub locking assembly is examined. Figure 10 shows the assembly with geometric feature tolerances applied.

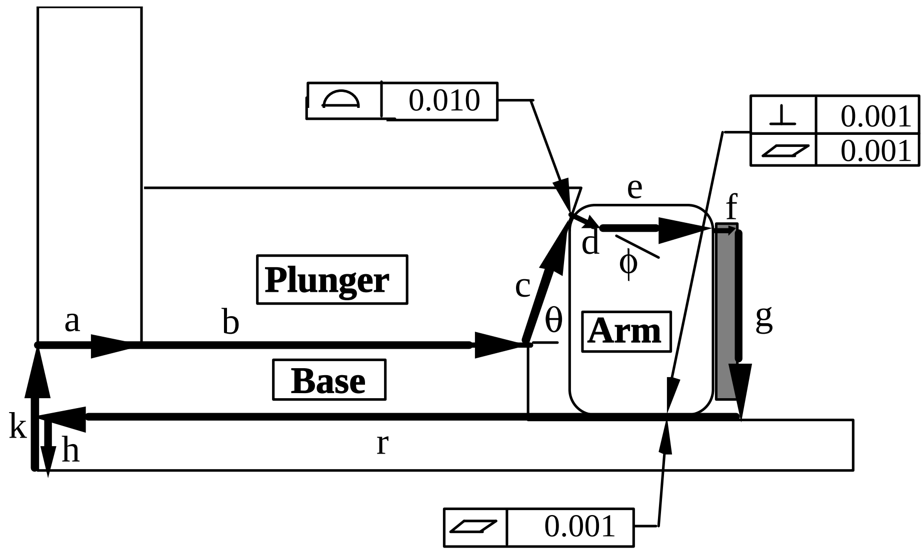

Figure 10. Tape hub locking assembly with geometric feature tolerances

In this case study, effort will be concentrated on the complete locking assembly model which is used to mount and hold a magnetic tape reel in place on a tape drive hub. The assembly consists of a plunger which can slide vertically against a guide mounted on the base and an arm which can slide horizontally on the base and which makes contact with the beveled surface of the plunger. The position of the arm is determined by the vertical position of the plunger, its level angle, the height of the arm, arm corner radius and the step on the base. As the plunger moves down, it forces the arm and pad outward against the inner surface of the tape reel, locking it in place.

The design requirement for the tape reel assembly is to have adequate interference between the pad and the tape reel. Then, the critical feature is the magnitude of the variation on dimension r when the tolerances of the components including geometric feature variations are considered.

Table 3 shows the geometric information for the assembly. Dimensions with a given tolerance are the manufactured variables, while those without specified tolerances are the assembly or kinematic variables. Table 4 lists all the geometric feature tolerances applied.

Table 3 Dimensional data of the tape hub locking mechanism

Part Name	Transformation	Nominal Dim	Tolerance(±)	Variation
Height I of base	Rotation	90°		
	Translation **k**	0.398	0.006	Independent
Width of base	Rotation	-90°		
	Translation **a**	0.355	0.005	Independent
Length of plunger	Rotation	0°		
	Translation **b**	1.000	0.01	Independent
Contact distance	Rotation	75°	0.5°	Independent
	Translation **c**	0.3194	?	Kinematic
Radius of arm	Rotation	-90°		
	Translation **d**	0.06	0.002	Independent
Width of arm	Rotation	15°	?	Kinematic
	Translation **e**	0.309	0.003	Independent
Thickness of pad	Rotation	0°		
	Translation **f**	0.05	0.002	Independent
Height of arm	Rotation	-90°		
	Translation **g**	0.488	0.004	Independent
Position of pad	Rotation	-90°		
	Translation **r**	1.854626	?	Kinematic
Height II of base	Rotation	90°		
	Translation **h**	0.203	0.002	Independent

Table 4 Geometric feature tolerance data of the tape hub locking mechanism

Name	Joint type	Feature	Tolerance band	Character length
α_1	Cylinder slider	Profile, arm corner radius	0.010	N/A
α_2	Planar	Flatness of arm	0.001	0.3690
α_3	Planar	Flatness of base	0.001	0.3690
α_4	Planar	Perpendicularity of arm	0.001	0.3690

The derivative matrices can be obtained according their definitions, i.e.,

$$[\,A\,] = \begin{bmatrix} \frac{\partial H_x}{\partial k} & \frac{\partial H_x}{\partial a} & \frac{\partial H_x}{\partial b} & \frac{\partial H_x}{\partial q} & \frac{\partial H_x}{\partial d} & \frac{\partial H_x}{\partial e} & \frac{\partial H_x}{\partial f} & \frac{\partial H_x}{\partial g} & \frac{\partial H_x}{\partial h} \\ \frac{\partial H_y}{\partial k} & \frac{\partial H_y}{\partial a} & \frac{\partial H_y}{\partial b} & \frac{\partial H_y}{\partial q} & \frac{\partial H_y}{\partial d} & \frac{\partial H_y}{\partial e} & \frac{\partial H_y}{\partial f} & \frac{\partial H}{\partial g} & \frac{\partial H_y}{\partial h} \\ \frac{\partial H_q}{\partial k} & \frac{\partial H_q}{\partial a} & \frac{\partial H_q}{\partial b} & \frac{\partial H_q}{\partial q} & \frac{\partial H_q}{\partial d} & \frac{\partial H_q}{\partial e} & \frac{\partial H_q}{\partial f} & \frac{\partial H_q}{\partial g} & \frac{\partial H_q}{\partial h} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 & 0.3980 & 0.9659 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1.3550 & -0.2588 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{26}$$

$$[\,B\,] = \begin{bmatrix} \frac{\partial H_x}{\partial c} & \frac{\partial H_x}{\partial f} & \frac{\partial H_x}{\partial r} \\ \frac{\partial H_y}{\partial c} & \frac{\partial H_y}{\partial f} & \frac{\partial H_y}{\partial r} \\ \frac{\partial H_q}{\partial c} & \frac{\partial H_q}{\partial f} & \frac{\partial H_q}{\partial r} \end{bmatrix} = \begin{bmatrix} 0.2588 & 0.6910 & -1.000 \\ 0.9659 & -1.4957 & 0.0000 \\ 0 & 1 & 0 \end{bmatrix} \tag{27}$$

$$[\,F\,] = \begin{bmatrix} \frac{\partial H_x}{\partial a_1} & \frac{\partial H_x}{\partial a_2} & \frac{\partial H_x}{\partial a_3} & \frac{\partial H_x}{\partial a_4} \\ \frac{\partial H_y}{\partial a_1} & \frac{\partial H_y}{\partial a_2} & \frac{\partial H_y}{\partial a_3} & \frac{\partial H_y}{\partial a_4} \\ \frac{\partial H_q}{\partial a_1} & \frac{\partial H_q}{\partial a_2} & \frac{\partial H_q}{\partial a_3} & \frac{\partial H_q}{\partial a_4} \end{bmatrix}$$

$$= \begin{bmatrix} 0.9659 & 0.2030 & 0.2030 & 0.2030 \\ -0.2588 & -1.8547 & -1.8547 & -1.8547 \\ 0 & 1 & 1 & 1 \end{bmatrix} \tag{28}$$

$$S^d = -[\,B\,]^{-1}[\,A\,]$$

$$= \begin{bmatrix} -1.0353 & 0 & 0 & -0.1457 & 0.2679 & 0 & 0 & 1.0353 & 1.0353 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -0.2679 & 1 & 1 & -0.3307 & 1.0352 & 1 & 1 & 0.2679 & 0.2679 \end{bmatrix} \tag{29}$$

$$S^\alpha = -[\,B\,]^{-1}[\,F\,]$$

$$= \begin{bmatrix} 0.2679 & 0.3717 & 0.3717 & 0.3717 \\ 0 & -1 & -1 & -1 \\ 1.0353 & -0.3918 & -0.3918 & -0.3918 \end{bmatrix} \qquad (30)$$

With the sensitivity matrix known, the variations of the kinematic or assembly variables can then be calculated by applying equation (9).

$$\begin{Bmatrix} \Delta c \\ \Delta f \\ \Delta r \end{Bmatrix} = \pm \begin{Bmatrix} 0.0082 \\ 0.5659^\circ \\ 0.0136 \end{Bmatrix} \qquad (31)$$

The variations of the kinematic or assembly variables without geometric feature tolerances is presented here for comparison.

$$\begin{Bmatrix} \Delta c \\ \Delta f \\ \Delta r \end{Bmatrix} = \pm \begin{Bmatrix} 0.0079 \\ 0.5000^\circ \\ 0.0124 \end{Bmatrix} \qquad (32)$$

In this assembly, dimension r is the one which has the design specification since its value and variation will affect the performance of the mechanism. If the design spec for r is set to be T_{ASM} and the estimated variation Δr is equal to 3 standard deviations, then the design spec can be used to calculate the assembly reject rate by either standard normal distribution table or integration or empirical methods.

9. Conclusions

A generalized approach has been presented for characterizing geometric feature tolerances in vector-loop-based assembly tolerance models and including such tolerances in the estimation of variations and assembly rejects for 2-D or 3-D assemblies. The method is simple and gives satisfactory results if the tolerances are relatively small compared with the associated nominal dimensions of the parts [Gao 1993].

In the example problem, the geometric feature tolerances increased the estimated assembly variations by about 10 percent (see equation (31) and (32)). This illustrates the importance of including geometric feature tolerances if an accurate estimation of assembly variation is desired.

Table 1 and Table 2 listed the most commonly occurring combinations of geometric feature variations and kinematic joint types in 2-D and 3-D assemblies. These tables are subject to change as new assembly conditions are considered or after more thorough studies are made.

Clearly, there are several problem areas requiring further study:

1. Characterizing geometric variations and their affects on mating part variations
2. Defining the interaction of geometric variations with size dimensions
3. Generalizing Rule #1 and its application to assemblies.

It is felt that vector models of assemblies constitute an efficient and effective tool for tolerance analysis of mechanical assemblies, which can be adapted as further model refinements are made.

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