New Tolerance Analysis Methods for
Preliminary Design of Mechanical Assemblies

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ABSTRACT

Two new methods have been developed to evaluate the effects of tolerance sensitivity and variation in mechanical assemblies. The methods are applicable to preliminary design, in which the dimensions and tolerances are only determined approximately.

The Variation Response Surface (VRS) combines design of experiments and response surface methodology, which are used so effectively for manufacturing process optimization, with statistical tolerance analysis. The VRS is a tool for early evaluation of the manufacturability of a design. An assembly function is required, relating the component dimensions to a critical assembly dimension. A mechanical assembly is represented as a point in a multi-dimensional design space, in which each axis represents the possible range of one of the component dimensions in the assembly.

The design space is sampled at a set of points. Critical assembly features are calculated at each point by linear and nonlinear analysis. A response surface is fit to the points. The surface can be fit of the mean, variance or sensitivity of the assembly variable. The surface may then be used to evaluate the design by searching in design space for the locations of the max and min values and by locating regions which are robust to variation (flat areas). The manufacturability of two competing designs may also be compared by constructing a design space for each and comparing Variation Response Surfaces.

The Variation Polygon is a new graphical representation for determining assembly tolerance sensitivities. It is analogous to the velocity polygon in kinematics, in which all the variations are brought to a common plane and added vectorially. Interactions between dimensional variations and the effect of changes in nominal dimensions show up clearly. In addition, all the tolerance sensitivities may be derived symbolically from sub-polygons. The symbolic and graphical nature of this tool provides considerably greater insight into the sources of variation than a simple numeric sensitivity value.
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Chapter 1

INTRODUCTION

Tolerances have long been recognized as an important factor influencing product quality. Statistical tolerance analysis of assemblies is a powerful method for predicting the effects of manufacturing variation on design requirements. However, it is generally only applied late in the design process, during detailed design checking. More efficient and informative methods are needed in the early design stages to help the decision-making process. This dissertation will present new methods for applying assembly tolerance analysis in the early design to assist in decisions of manufacturability and cost. Procedures for creating tolerance analysis models during preliminary design, using approximate dimensions, will be demonstrated, along with a systematic method for examining a broad range of the design space to search for regions which are robust to variation.

1.1 MOTIVATION

In most industrial practice, the quality control procedure on the production floor is normally referred to as on-line control. Engineers are needed to monitor the production line very closely to assure that variation does not exceed quality limits. It is labor-intensive and costly. This is downstream quality control, because it takes place during final production. In contrast to strict downstream quality control, many companies think a better way to assure quality is to move control upstream, to the design stage, which would be less expensive and more effective. This shift of emphasis reflects the evolution of quality control from inspection and process control to design improvement, i.e., to designs which are more robust to manufacturing variation. By moving manufacturing decisions upstream, significant reductions in both design and manufacturing cost can be realized, as well as product lifetime cost.

Manufactured parts are not generally used individually, but as assemblies of parts. Each part must mate with others correctly in order for the assembly to function properly. However, variations which occur in manufacturing, due to imperfection of the processes, can cause a mechanical assembly to depart from the specified design requirements. Tolerance limits must therefore be set on part dimensions to assure "good" assemblies are produced.

For the designer, the emphasis is usually on fit and function, which affects performance. So, tight tolerances are preferred. For manufacturing, loose tolerances are
usually preferred for ease of manufacturing. These competing requirements can produce tension between them. If the designer assigns tolerances tighter than necessary, the cost of the product will be very high. On the other hand, even though relaxed tolerances make manufacturing easier to handle, the rejects, rework, and possible redesign and process changes due to out-of-specification assemblies will push the cost higher again. Since the cost is in terms of the lifetime of the product, appropriate selection of tolerances is critical for the company.

The following figure shows tolerance as a link between engineering design and manufacturing, as well as the consequences in each step.

![Diagram of A Critical Link Between Design and Manufacturing](image)

Figure 1.1 A critical link between design and manufacturing

Tolerances are the bridge between design and manufacturing. They play an important role as the connection between cost, quality and performance. Tolerance analysis can provide a common ground for communication among design and manufacturing personnel, and help to produce much higher benefits for the company.

Tolerances are related to producibility and manufacturability. Traditionally, tolerance analysis is performed after the design is completed. The push of tolerance analysis upstream will assure quality products and bring great economic benefits, as the cost of design changes is much less when changes are made at an early stage of the design process. Tolerance analysis methods will make possible better design for quality, so that the products are more "tolerant" of variation due to manufacturing.
Therefore, tolerance analysis in the early design stage is very important, and will receive greater emphasis in successful companies. Methods to facilitate early tolerance evaluation need to be developed. They will allow designers to evaluate assembly quality in a preventative manner. The economic effect will be tremendous.

### 1.2 THE NEED FOR EARLY TOLERANCING METHODS

There are different methods for considering variation in design. Tolerance analysis methods include: Worst Case analysis, Linearized analysis (Taylor-series method), Estimated Mean-shift, Robustness Optimization, Taguchi Method, Monte Carlo Simulation, Numerical Integration (quadrature method), Method of System Moments, Reliability Index, and Response Surface Technique. All of them except Worst Case use a statistical approach [Chase et al. 1991].

Even though much work has been done on tolerance analysis, there are still some points not clearly addressed. For most of these methods, estimates of manufacturing variation data or statistical distribution parameters of the design dimensions need to be provided for the completed design. Tolerance analysis is then performed and the results are compared with the design specifications. Such tolerance analysis is normally performed after dimensional synthesis.

The early design, or preliminary design, is often called “conceptual design” as there are still major decisions to be made [French 1985]. The design process is iterative by nature. The design process can not solely depend on intuition, experience, or even luck as the complexity of the design and number of specific requirements increase. The structure and dimensions in the design will be modified many times in the design stage based on the synthesis and the analysis results. How well the tolerance analysis method can be combined with such changes will decide the efficiency and the effectiveness of the design.

Conceptual design requires qualitative methods for making comparisons. Most methods used now provide quantitative information in the form of data analysis for a specific design. However, such an analysis provides results for only one design. Any possible design change will require the designer to repeat the complete analysis. Not only quantitative, but also qualitative information is needed by the conceptual designer. If both quantitative and qualitative information are provided in parallel for the design, their functions will complement and compensate each other.
The methods must also convey qualitative information about assembly relationships, such as the sensitivity of each dimensional variation to assembly tolerance requirements or the influence of small adjustments between mating parts on assemblability. If tolerance relationships can be obtained, the designer can be freed from repeatedly building and analyzing engineering models. The extensive usage of tolerance information in the early design stage will greatly help designers concentrate on higher level design architecture. Specific cases can be drawn from general relationships.

Also, the methods for evaluating tolerance information need to be extended beyond the analysis of a single design point to a range of points in design space to help the designer have a better grasp of the design quality. Qualitative information like trends and relationships will be seen more clearly in the early design stage. The possibility of evaluating the robustness of a design to variation or searching design space for a region exhibiting less sensitivity to variation will assist the designer in assessing manufacturability of a design. Alternative designs can readily be compared in the early design stage, before investing in a detailed design. It will facilitate design by permitting the integration of the synthesis process with tolerance analysis, instead of the traditional approach of putting error analysis as the post synthesis activity.

1.3 RESEARCH OBJECTIVES

The objectives of this research are to identify and develop an understanding of the mechanism governing tolerance sensitivity and variation in mechanical assemblies, and to develop tolerance analysis methods which will be suitable for early design stages. Desirable features of the methods are listed as follows:

1). The methods should include both qualitative and quantitative analyses of tolerance sensitivity and variations.

2). The tolerance information will be easily obtained for the successive nominal changes which are common in the early design stage.

3). Alternative designs may be compared and evaluated.

4). They should emphasize the relationship among the nominal dimensions, variations, tolerance sensitivities.

5). They should define the relationship between the independent and the dependent variables in the assembly, rather than only specific data values.
They should simplify the analysis and encourage alternative designs.

1.4 RESEARCH APPROACHES

Surface Approach

Three surface methods will be investigated for assembly tolerance analysis when manufacturing variations are considered. The proposed methods will include variation response surface, quadratic variation surface and real variation surface. The design space shall be defined by the acceptable range of design variables and constrained by assembly closure requirements. The response surface and vector-based linearized tolerance analysis methods will be combined to produce surfaces representing the variation of critical assembly parameters. The quadratic variance surface will be derived from the quadratic nominal surface estimation. The real surface will be calculated by combining the nonlinear and linearized methods at each design point and integrating them into the optimization process. Case studies will be analyzed by each surface approach and comparisons will be made.

Variation Geometry Approach

Tolerance sensitivities can be calculated by numerical methods, but it prevents a complete understanding of the nature of tolerance sensitivity. Data result is only associated with a single design. The nature of the sensitivity will be investigated for the mechanism of kinematic adjustment within an assembly. This study will include geometric or analytic analyses for tolerance analysis of mechanical assemblies. The study will focus on the relationship among tolerance parameters: independent and dependent variations, tolerance sensitivities, and kinematic adjustments, as well as representing them graphically and analytically. Methods providing both qualitative and quantitative information about assembly variations will be developed. The analogy between dimensional variation and velocity analysis will be investigated. Several assemblies will be analyzed to give a spectrum of case studies. The geometric properties will be defined and summarized in the process.

1.5 OVERVIEW OF DISSERTATION

This chapter discussed the motivation, the need for early tolerancing methods and the research objective. The next chapter presents a literature review of related research. Current tolerance analysis methods, robust design and other design strategies will be
reviewed. The limitations of each will be discussed. Chapter 3 presents new surface methods for tolerance analysis. The case studies and comparison for these methods will be in Chapter 4. Chapter 5 describes the nature of sensitivity and develops a new effective method for tolerance analysis: variation polygon for evaluating tolerance sensitivity and assembly variations. Chapter 6 discusses the analogy between variation and velocity analysis and gives more applications of the variation polygon. More analyses of examples from variation geometry including variation polygon, pseudovectors, frame polygon and projections are presented in Chapter 7. Conclusions and recommendations for future work are included in Chapter 8. References will conclude the dissertation.
Chapter 2

LITERATURE REVIEW

This chapter reviews possible conceptual design methods along with different approaches for considering variation in design. The conceptual methods include type and dimension synthesis, network graphs and topological synthesis. Variational methods include the vector-based method, robust design, response surface method, variational geometry and error sensitivity. The limitations of preliminary tolerance analysis will be discussed.

2.1 CONCEPTUAL DESIGN

Conceptual design is a creative process by which numerous alternative solutions are generated and considered without detailed analysis. Each concept is presented by sketches, schematics or other simple representations and evaluated qualitatively. The focus is on defining the basic elements functionally, generating several candidate systems for performing each function and investigating the feasibility of various combinations and configurations. Various criteria may be used in evaluating feasibility, including estimated performance, practicality, manufacturability, cost, market acceptance, etc.

As currently practiced, conceptual design has not emphasized the effects of dimensional variations on the design. These are directly related to manufacturability and cost, which are important effects to include in evaluation of design in the early-design stages. Manufacturing consideration can eliminate costly change later on.

Several conceptual design methods have been developed for mechanism design and will be presented here in that context. Because mechanical assembly may be treated as a mechanism with zero degrees of freedom, we can apply the methods of mechanism design to mechanical assemblies that would not typically be considered mechanism. Mechanism design at the conceptual level includes several approaches to synthesis: type synthesis, dimensional synthesis and graph analysis.

Type Synthesis

Type synthesis is the process of determining several types of mechanisms for performing a desired function. It involves creation of possible "equivalent" mechanisms, without regard to the dimensions of the components. For example, consider the problem of transmitting rotary-to-rotary motion. Possible mechanisms include a belt and pulleys,
chain and sprockets, a gear pair, friction disks, a pair of servo motors, a hydraulic pump and motor, fluid or magnetic couplings, dual u-joints and shaft, a parallelogram linkage, etc. For partial rotation, the list could also include cams, four-bar linkage, gear segments, etc.

**Dimensional Synthesis**

Type synthesis considers the "structure" or "architecture" of the design, while dimensional synthesis determines the "proportions" [Vucina et al. 1991] or values of the design parameters [Kota et al. 1992]. Dimensional synthesis is used to determine the geometric characteristics of the links which would allow the selected mechanism to perform the desired task. It could include kinematic synthesis, kinematic analysis and dynamic analysis. Assignment of different dimensions could produce different motion characteristics even for a simple kinematic chain. The component dimensions of the given type of mechanism must be determined and evaluated. These dimensions must satisfy the functional requirements and constraints, including dimensional, static and dynamic.

Design refinement would be the next step after a mechanism is selected and approximate dimensions are determined. Sensitivity analysis, optimization based on different criterion, such as the minimum structural error in function generation, etc., would be performed for quantitative design. Then the tolerance due to manufacturing error would need to be analyzed, as it would influence the kinematic and dynamic behavior [Rothbart 1964].

Dimensional synthesis can be approximate or exact, geometric or analytic. By employing approximate geometry and design parameters, it may be used for conceptual design. Or, it may be used for detailed quantitative design during design refinement. Geometric methods could give direct feeling for mechanical detail, while analytical methods would provide economical and practical solutions [Hartenberg et al. 1964].

**Graph Analysis**

**Graph Representation**

The application of graph theory (the topology of networks) to structural analysis of mechanical systems shows that mechanisms and mechanical systems have the topological or network property [Dobrjansky et al. 1967]. Graph theory provides one way to represent mechanisms topologically. The "structure" or "topology" of a mechanism is defined by the
number of links, number and types of joints connecting links, and which links have been selected as the ground and input. Type synthesis includes topological synthesis and analysis [Woo 1967, Olson et al. 1985].

In a network graph of a mechanism, each link is represented by a vertex and each joint or kinematic pair connecting two links is represented correspondingly as an edge connecting two vertices in the graph. The degree of the vertex indicates the number of joints connecting corresponding links [Olson et al. 1985]. Matrices of incidence represent the graph mathematically.

Kinematic chains are also used to represent mechanisms abstractly. A kinematic chain is a closed chain containing rigid links, which are connected by simple lower pair joints, and no ground link. In this mode, each vertex represents a joint and each edge represents a side of a link. The Watt six-bar chain and its graph are shown in Figure 2.1.

![Figure 2.1 Watt chain and graph](image)

The edge-vertex incidence matrix corresponding to the graph of a Watt chain in Figure 2.1 is:

\[
\begin{array}{cccccccc}
A & B & C & D & E & F & G \\
1 & 0 & 0 & 1 & 0 & 0 & 0 \\
2 & 1 & 1 & 0 & 0 & 0 & 0 \\
3 & 0 & 1 & 1 & 0 & 1 & 0 \\
4 & 0 & 0 & 0 & 0 & 1 & 1 \\
5 & 0 & 0 & 0 & 0 & 0 & 1 \\
6 & 0 & 0 & 1 & 1 & 0 & 0 \\
\end{array}
\]

(2.1)

The columns represent the edges, or joints, in the network graph as A, B, C, D, E, F, G. The rows represent the vertices, or parts: 1, 2, 3, 4, 5, 6. A "1" in the ij element means the j edge is incident to the i vertex. The two ones in column A, above, means that joint A connects links 1 and 2.
The incidence matrix represents the topological information such as connectivity. It is a mathematical representation of the network graph. Several methods for identifying isomorphic graphs have been based on the incidence matrix [Dobrjansky et al. 1967]. The incidence matrix has also been successfully used for automatic generation of kinematic chains and corresponding kinematic equations for a mechanism [Larsen, 1991].

Other properties of network graphs were defined by Wilson [Wilson 1985]:

"A planar graph is one which is isomorphic to a plane graph. A plane graph is a graph drawn in plane in such a way that no two edges intersect geometrically except at a vertex to which they are both incident."

Euler's equation may be used to find the number of independent loops from the network graph.

For a plane graph

\[
V - E + A - K = 1, \tag{2.2}
\]

here:  \( V \): number of vertices \\
\( E \): number of edges \\
\( A \): number of faces \\
\( K \): number of components \\

or in expression

\[
\#\text{vertices} - \#\text{edges} + \#\text{faces} - \#\text{components} = 1, \tag{2.3}
\]

For a connected plane graph, component K is one. Euler's formula would be

\[
V - E + A = 2, \tag{2.4}
\]

or in expression

\[
\#\text{vertices} - \#\text{edges} + \#\text{faces} = 2 \tag{2.5}
\]

Faces are defined by Jordan curve (composed of independent circuits and infinite face), so L (the number of loops) will be introduced.

\[
\#\text{vertices} - \#\text{edges} + (\#\text{loops} + 1) = 2, \tag{2.6}
\]

\[
\#\text{loops} = \#\text{edges} - \#\text{vertices} + 1 \tag{2.7}
\]
For connected plane graphs, the Euler theorem states that

\[ L = E - V + 1 \]  \hspace{1cm} (2.8)

For the Watt mechanism shown in Figure 2.1, the number of the loops, as calculated by equation (2.8), is two. One set of independent loops is shown in the graph.

**Mechanism Loop Generation**

Graph representation may be used to create vector loops or kinematic chains. When a network or kinematic graph is created for a mechanical assembly, the vertices in the graph represent the parts. All the dimensional and form information of the manufactured parts are eliminated. The edges of the graph represent connecting joints or contact points, which include the mating information and the kinematic variation due to manufacturing variation. A network graph is only an abstract topological representation, so additional methods are needed to describe the geometric data and the relationships.

The vector loops describing a mechanism are not unique. There are many ways to create the loops to pass through all the parts and the joints. However, the total number of independent loops for one assembly is unique [Paul, 1963]. If the kinematic graph is a plane and connected graph, the number of independent closed loops can be obtained from following Euler’s theorem

\[ \#\text{loops} = \#\text{joints} - \#\text{parts} + 1 \]  \hspace{1cm} (2.9)

The transformation to graph representation highlights the connectivity and makes loop generation more clear. Most of the time the loops may be found by inspection.

The planar or non-planar form of a graph has nothing to do with the distinction between planar and spatial kinematic chains [Olson 1985].

Freudenstein [Freudenstein et al. 1979] stated that the advantages of graph representation of mechanisms include:

1. Network properties of graphs are directly applicable to mechanisms.
2. Unique identification of kinematic structure can be made.
3. A single atlas of graph structure can be used to enumerate mechanisms.
4. It leads to the creation of mechanisms by separation of structure and function.
5. It allows for automatic kinematic and dynamic analysis of mechanisms.
Topological Synthesis and Analysis

Designers may use topological synthesis to enumerate all nonisomorphic basic kinematic chains corresponding to a specified number of links and joints and given degrees of freedom. Then, the designer can enumerate all the possible choices for the ground link for the set of mechanisms. This enumeration process is independent of functional requirement considerations. Sometimes it is called "structural synthesis" or "number synthesis."

For the case of kinematic chains composed of eight links or less and simple hinge joints with one degree of freedom, Crossley used graph theory to produce complete permutation maps, which in turn tell the number of different kinematic chains possible [Crossley 1965].

For planar mechanisms with N links, joined by J hinged joints and with one link fixed, the degrees of freedom F may be calculated from Grubler's equation

\[ F = 3(N - 1) - 2J \]  
\[(2.10)\]

If there are multiple joints (a joint connecting more than two links), or redundant joints (joints acting in parallel), care must be taken to count the correct number of the joints.

If the degrees of freedom equal one, it defines a mechanism where

\[ 3N = 2J + 4 \quad \text{in the kinematic chain} \]  
\[(2.11)\]

The number of the links and joints would be constrained as follows:

N must be even,
N = 4, J = 4
N = 6, J = 7
N = 8, J = 10

The number of independent loops L is related to the number of links by

\[ L = 1/2N - 1 \quad \text{(for F = 1)} \]  
\[(2.12)\]

In the topological representation, equation 2.11 becomes:

\[ 3V = 2E + 4 \]  
\[(2.13)\]
The topological graph contains a set of vertices, certain of which are joined together in a network by edges.

Equations (2.13) and (2.8) would be used to do a complete permutation.

The basic graph equations describing kinematic structures are [Freudenstein et al. 1979, Vucina et al. 1991]:

\[ 2J = \sum \limits_{i} \text{i} \ N_i \]  
\[ L = J - N + 1 \]  
\[ F = \sum \limits_{i} f_i - \lambda (J - N + 1) \]  

Combining (2.15) and (2.16)

\[ F = \sum \limits_{i=1}^{J} f_i - \lambda L \]  

Here:  
\text{N}_i: \text{the number of links with i joints}  
\text{f}_i: \text{degrees of freedom of ith joints}  
\text{F}: \text{degrees of freedom of mechanism}  
\lambda: \text{degrees of freedom of the space within which the mechanism operates.}  

For plane motion \( \lambda = 3 \), for spatial motion \( \lambda = 6 \).

Vucina stated that the number of the loops in a graph may be regarded as a measure of complexity of a corresponding mechanism.

The following shows another way to do the enumeration. From the left to right, by adding degree-two vertices to a contracted graph, the graphs of Watt and Stephenson chains are obtained as the two possible six bar mechanisms.

![Figure 2.2](image)

Figure 2.2 Contracted graph, graphs of Watt and Stephenson kinematic chains
Comparing Figure 2.1 and Figure 2.2, it is seen that a graph is only a topological (network) representation. The dimensions measured from the network graph have no meaning at all in terms of the component dimensions. At this stage no attention is paid to dimensions or other geometry except connectivity. Isomorphic structures need to be eliminated. Different enumeration methods have been developed [Mayourian et al. 1984, Tuttle et al. 1989]. Automatic sketching algorithms have been demonstrated for transferring the graph to a line graph for kinematic chains [Olson et al. 1985\footnote{2}] or for mechanisms [Wu et al. 1988], or from a combinatorial approach [Chieng et al. 1990].

Topological analysis would follow synthesis procedures to determine a distinct way to decide input, output, joint type, etc. Assigning different joint types would create different mechanisms [Freudenstein et al. 1979]. Figure 2.3 shows two examples of a Watts chain in the assigned graph representation. The joints of the graph are assigned as revolute (pin) and prismatic (slide) types. There is still no dimension information.

![Figure 2.3 Assigned graphs](image)

Possible Watt mechanisms corresponding to Figure 2.3 are shown in Figure 2.4.

![Figure 2.4 Watt mechanisms: pump and window-awning guide](image)

Olson stated "The evaluation of mechanism topologies based on the functional requirements is the least systematic step" [Olson et al. 1985\footnote{1}].
2.2 VECTOR-BASED METHOD

Vector mathematics has been used for kinematics and dynamics of machinery for many years [Suh et al. 1978, Chace 1983, Erdman et al. 1984]. A mechanism, such as a robot or a packaging machine, is described by vector chains or loops representing the physical dimensions, connected by kinematic joints describing the relative motion between members. Prescribed motions are defined for certain members and the resulting motions of the remaining members are determined by solving the vector loop equations.

A similar vector-based method has been used to model variation in mechanical assemblies and for tolerance analysis [Marler 1988, Chase et al. 1987, Chase et al. 1994]. An assembly is treated as a kinematic mechanism, described by vector loops and kinematic joints. However, it differs fundamentally from classical kinematics, which uses rigid bodies of fixed dimensions and large displacement kinematic inputs. The inputs to the variational model are small changes in the dimensions due to manufacturing variations. The outputs are small kinematic adjustments between mating parts in response to the dimensional variations.

Velocities and accelerations are not considered. The variations are not time-dependent. They represent the resulting manufacturing variations from one assembly to the next as parts are selected from bins and assembled.

Since dimensional variations are small compared to their nominal values, the solution is obtained by linearizing the loop equations and applying linear algebra to obtain solutions for small variations in assembly parameters about the nominal.

Tolerance analysis is performed by summing the component variations statistically or by worst case. The resulting estimates of assembly variation are compared to the specified design limits and the percent rejects may be predicted statistically.

The vector-based method is introduced in the following section.

Assembly Modeling

In a vector loop model of an assembly, kinematic joints describe the degrees of freedom of motion between mating parts. The vector loops describing the assembly need to be produced according to a set of modeling rules [Chase et al. 1992]:

1. At least one loop must pass through every part and every joint in the assembly.
2. No single loop may pass through a part or joint more than once, but it may start and end at the same point.

3. There must be enough loops to solve all the kinematic variables, one loop for every three variables in 2-D assemblies or for every six variables in 3-D assemblies.

Other rules apply to the path of a loop through specific kinematic joints, locating joints with respect to datum reference planes, etc. Geometric form and feature variations, as defined by ANSI standard Y14.5 [ANSI 1982], may also be inserted into the vector loops [Robinson 1989, Chase et al. 1989, Ward 1992].

Thus, the assembly relationships between the dependent and the independent variables are expressed implicitly. The loop equations, i.e., residual equations are used as generalized equations for the assembly. They act as assembly constraint functions.

**Linearized Solution**

The linearized solution method is outlined below. Geometric variations will not be considered in this presentation, as they are analyzed in a very similar way. By Taylor's series expansion about the nominal assembly dimensions, the differential of the loop functions may be written in terms of their partial derivatives and expressed in matrix form:

\[
\{dH\} = [A]\{dX\} + [B]\{dU\} 
\]

(2.18)

\{dH\}: differential of loop equations or the variation of assembly functions,

\{dX\}: variations of the independent variables or manufacturing dimensions, which are specified as symmetric component tolerances,

\{dU\}: unknown variations of the dependent variables or assembly resultants, which need to be estimated by kinematic analysis. Dependent variables adjust kinematically during assembly,

[A]: first order partial derivative matrix of loop functions with respect to the independent variables (manufactured dimensions, X),

[B]: first order partial derivative matrix of loop functions with respect to the dependent variables (assembly variables, U).

[A] and [B] are the geometric sensitivity matrices for closed loop constraints. The derivatives need to be evaluated at their nominals. For planar assemblies, [A] is a \(3p \times m\)
matrix (m stands for the number of independent variables, p stands for the number of independent loops), and \([B]\) is a \((3p \times 3p)\) square matrix, for the determined system, where the number of the dependent variables equals three times the number of loops.

Closed Loop

For the closed loop assembly, the residual equals zero for length equations and equals a multiple of \(2\pi\) for angle equations. The dependent variations \(\{dU\}\) can be solved from Equation 2.18 as:

\[
\{dU\} = -[B]^{-1}[A]\{dX\} \tag{2.19}
\]

or written as

\[
\{dU\} = [S_c]\{dX\} \tag{2.20}
\]

The \(\{dU\}\) is the weighted summation of the independent variation \(\{dX\}\). The weight term, i.e., matrix \(-[B]^{-1}[A]\), is referred to as the tolerance sensitivity matrix for closed loops. It is expressed as \([S_c]\) in Equation 2.20. It is the partial derivative of the dependent variable \(U\) with respect to the independent variable \(x_i\) or \(\frac{dU}{dx_i}\) and evaluated at the nominal value. It describes the relationship between variations of the dependent and independent variables.

Open Loop

The open loop case is similar to the closed loop analysis, but the vector loop describes a nonzero resultant vector or feature \(H_0\), representing a gap or critical assembly clearance. The residual \(\{dH_0\}\) is also nonzero for open loops, representing the variation in the resultant open loop vector. Clearance, orientation, position or other feature variations may be represented by open vector loop equations. The differential expression is:

\[
\{dH_0\} = [C]\{dX\} + [D]\{dU\} \tag{2.21}
\]

\(\{dH_0\}\): differential of the open loop equation, or variation of the clearance or dependent assembly variables,

\([C]\): first order partial derivative matrix of the open loop function with respect to the independent variables, \(X_i\),
[D]: first order partial derivative matrix of open loop function with respect to the dependent variables, $U_i$.

[C] and [D] are the geometric sensitivity matrices for the open loop constraints.

The $\{dU\}$ can be expressed in terms of the closed loop results, i.e., equation (2.19). So

$$\{dH_0\} = ([C] - [D][B]^{-1}[A]) \{dX\}$$  \hspace{1cm} (2.22)

The matrix $([C] - [D][B]^{-1}[A])$ is referred to as the tolerance sensitivity matrix for open loops or for assembly resultants. It is represented as $[S_O]$ or $\{dH_0/dx_j\}$.

**Tolerance Analysis**

Various methods for estimating tolerance accumulations in mechanical assemblies are presented in Table 2.1. Tolerance sensitivity matrix $[S]$, either $[S_o]$ or $[S_c]$, is used differently by the various criteria, which are based on the design requirement [Chase et al. 1991]. The dependent variations $dU$ are estimations of the variation of critical assembly features. They are compared with the specified tolerance $T_{asm}$.

<table>
<thead>
<tr>
<th>Criterion</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Worst Case</td>
<td>$dU = \sum</td>
</tr>
<tr>
<td>Root Sum Square</td>
<td>$dU = \sqrt{\sum S_i^2 , dx_i^2} \leq T_{asm}$</td>
</tr>
<tr>
<td>General Root Sum Square</td>
<td>$dU = C_f Z \sqrt{\sum S_i^2 \left( \frac{dx_i}{Z_i} \right)^2} \leq T_{asm}$</td>
</tr>
<tr>
<td>Motorola Six Sigma Model</td>
<td>$dU = Z \sqrt{\sum S_i^2 \left[ \frac{dx_i}{3C_{pi}(1-m_i)} \right]^2} \leq T_{asm}$</td>
</tr>
<tr>
<td>Estimated Mean Shift Model</td>
<td>$dU = \sum m_i \left(</td>
</tr>
</tbody>
</table>

$T_{asm}$: design limit for variation $\{dU\}$, i.e., allowable limit for variation of dependent variables or assembly resultants. It is assigned by the designer. It needs to be satisfied in the design.
Cf: correction factor to account non-ideal condition, e.g. non-normal distribution. It ranges from 1.4 to 1.8,

Z, Z1: number of standard derivations corresponding to the assembly and component tolerance limit,

Cpi: process capability ratio,

m1: mean shift factor. It is in the range of zero to one. Zero m1 is used for short term variation in the process and stands for statistical case.

Worst case analysis assumes all the fluctuations of the components occur simultaneously at worst combination, even though it is most unlikely in an assembly process. It predicts zero rejects. The assigned tolerance for each component would be too tight to be manufactured realistically if many parts are involved. It is the most conservative estimation and it is unrealistic.

RSS, general RSS and Six Sigma are statistical tolerance accumulation models and predict non-zero rejects. They assume a normal distribution for component variations and the resultant variation of the accumulated tolerance. RSS uses ±3σ component tolerance limits to predict the rejects. General RSS allows other than ±3σ normal distribution. Statistical tolerance analysis assigns looser component tolerances than worst case.

The Motorola Six Sigma Model distinguishes between short term and long term process capability, which quantifies the spread of the process. It includes the mean shift factor m1. Cpi and m1 are chosen to account for the degree of uncertainty in individual process characteristics.

The estimated mean shift model estimation will fall between the worst and statistical case.

The foregoing steps have described vector-based methods for tolerance analysis. Efforts are being made to integrate vector-based systems with commercial CAD systems. Using such a system, a model of an assembly can be built upon a CAD model and used to make quantitative estimates of the effect of manufacturing variations on assembly performance. The connection between CAD and tolerance analysis eliminates the duplication of the data entry for the geometry, avoids possible errors, and speeds up the
analysis procedure. The tolerance analysis model is created graphically and stored with the CAD model, along with all the assembly relationships and data required for analysis.

2.3 ROBUST DESIGN

One of the objectives of conceptual design is to evaluate the proposed design to see if it is robust to variation. By applying tolerance analysis in the early design stages, the effects of variation on performance can be determined. Robust design also requires information on how variation changes with a change in the nominal dimensions. Several methods for improving design robustness were considered.

Feasible Robustness

Design with optimum and robust performance involves incorporating variations and uncertainties due to manufacturing. The feasible robustness approach accounts for the transmitted variation to make the design have robust optimum performance [Balling et al. 1986, Parkinson et al. 1992, Sundaresan et al. 1993]. The procedure is described as follows.

Problem Definition

The problem is defined as:

$$\text{Max. } f(x,p)$$

subject to:

$$g_i(x, p) \leq b_i \quad i = 1, ..., m.$$  \hspace{1cm} (2.23)

$$L \leq x \leq U$$

Here:

- $x$ stands for the independent variables, i.e., controllable variables. It is a n-dimensional vector with $L$ as low limit and $U$ as upper limit.
- $p$ stands for the uncontrollable parameters. It is a l-dimensional vector.
- $b_i$ are inequality constraints, and they are grouped into a vector $b$.

Transmitted Variations

In robust optimization, the effect of the variation needs to be considered for the feasibility of the optimum design. The fluctuation of the $x$, $p$ and $b$ about the nominal
value will transmit the variation into functions. The linear model is used to analyze the effect of the tolerance.

The transmitted variation can be examined by worst case analysis and statistical analysis.

For worst case

\[
\Delta g_i = \sum_{j=1}^{n} \frac{\partial g_i}{\partial x_j} \Delta x_j + \sum_{j=1}^{1} \frac{\partial g_i}{\partial p_j} \Delta p_j
\]

(2.24)

For statistical analysis

\[
\sigma_{g_i}^2 = \sum_{j=1}^{n} \left( \frac{\partial g_i}{\partial x_j} \sigma_{x_j} \right)^2 + \sum_{j=1}^{1} \left( \frac{\partial g_i}{\partial p_j} \sigma_{p_j} \right)^2
\]

(2.25)

Worst case assumes the all the variations occur at the same time. It is considered the most conservative estimate. Statistical analysis assumes all variations are independent and random.

For feasibility, the constraint function is modified by the transmitted variation. They include the effect of the variation of the design variables as:

For worst case

\[
g_i(x,p) + (\Delta b_i + \Delta g_i) \leq b_i
\]

(2.26)

For statistical analysis

\[
g_i(x,p) + k \sqrt{\sigma_{b_i}^2 + \sigma_{g_i}^2} \leq b_i
\]

(2.27)

Values of k are based on the area in a single tail of a standard normal distribution, since it is only fluctuations in one direction that can result in a constraint violation. It would be chosen by the designer. If the constrained variation is normally distributed, a certain k means that the currently binding constraints would still be satisfied at a corresponding percent acceptance value for large number of fluctuations. A bigger k means a higher percentage of feasible designs.

The following plot shows the effect of the transmitted variation.
Figure 2.5 Transmitted variation approach for robust design

The robust optimum is the optimum point in the modified constrained feasible region. The decrement of the feasibility region makes the design more "tolerant" of variations. In other words, the new optimum at the boundary of the reduced feasible region is the robust optimum under variations.

Taguchi Method

Traditionally, engineers conduct sensitivity analysis after design optimization. Taguchi introduced parameter design to improve the quality of products whose manufacturing process involves significant variability or noise [Taguchi et al. 1989]. To make a design robust to variation means fewer rejects will be produced. The design can perform properly over a wider range of variation.

The Taguchi method is a statistical experiment method. This method is able to examine more factors in a small number of experiments through the use of the orthogonal array approach, i.e., fractional factorial design. The loss function and signal-to-noise ratio are some of the new concepts used. Minimizing the loss associated with product becomes the objective for the engineer [Lin 1987]. Product quality is a measure of the social loss of the product. By Taguchi's definition, it includes losses to both customers and producers.
Three design steps are applied by Taguchi, as design has greatest impact on product quality. They are system design, parameter design and tolerance design. The products for each stage are:

System design: a basic functional prototype design.

Parameter design: a set of nominal parameters which minimizes the performance variation. This set of parameters describes the design output as "signal" and performance variation due to manufacturing variation as "noise." This signal-to-noise ratio approach improves the quality of the product. It reduces effect of the "noise" by reducing the sensitivity of engineering design to sources of variation rather than controlling the source. The variation sources are often uncontrollable, such as from the environment, or manufacturing imperfections.

Tolerance design: tolerances about the parameter setting which minimize product cost. A trade-off between quality loss and cost is needed.

The following plot shows the idea of the sensitivity reduction method through parameter design.

![Diagram](image)

Figure 2.6 Sensitivity reduction by shifting nominal value
The quality level of products is evaluated by using loss function approach for three types of tolerances:

- The-Nominal-The-Better (N type),
- The-Smaller-The-Better (S type),
- The-Larger-The-Better (L type).

A N type tolerance is required when nominal size is preferred, such as diameter of shaft. Bilateral tolerances are usually used in this type of application. A S type tolerance involves a nonnegative characteristic, whose ideal value is zero, such as impurity, wear or deterioration. The-Larger-The-Better is applicable to characteristics such as fuel efficiency [Taguchi et al. 1989].

The Taguchi method is a cost effective method [Kackar 1985]. A modification of the method by Gaussian quadrature is described in terms of higher-order statistical moments [D'Errico et al. 1988].

**Response Surface Method**

The response surface method is a statistical method [Box et al. 1987]. It is used to analyze system behavior where a large number of variables influence some feature of the system. The procedure is a collection involving experiment strategy, mathematical methods and statistical inference, which enable the experimenter to make empirical exploration of the system [Myers 1971].

Beard [Beard et al. 1993] has used the loss function concept in Taguchi quality engineering technique to analyze suspension design. The loss function, which characterized the performance of the product, jointly considers the mean and the variation. The response surface method has been used to consider the effect of linear and quadratic terms. The expected loss is plotted in terms of the two design variables. The plot shows the trend of the loss function over a region of feasible designs and gives designers further suggestion for robust design.

**2.4 VARIATIONAL GEOMETRY SYSTEM**

Variational geometry is a method for the early or "conceptual" phase of mechanical design, and appears very attractive. The system allows the designer to interactively sketch unscaled, e.g., undimensioned or partly dimensioned geometry, and constrain that geometry by linking it to governing equations. The dimensions and other variables derived
must satisfy the constraints [CIME 1989]. The variational geometry system is often referred to as a dimension-driven system because the new geometry is derived from the constrained dimensions [Martino et al. 1989].

The variational geometry has been used in design to create mechanical parts [Lin et al. 1981, Light et al. 1982]. "In the variational geometry, higher order entities (edges, surfaces and dimensions) are defined with respect to characteristic points." The algebra equations relate the characteristic points to the constraints. Constraints are imposed to define geometry. The constraints are described as implicit or explicit. Parallel plane, perpendicular plane and plane defined by three non-collinear points are examples of implicit constraints. Distance between two points and angle between two lines are examples of explicit constraints. The explicit constraints are dimensional constraints since they limit the location of the characteristic points. They are expressed in terms of analytical geometry.

For example, P1 (x1, y1, z1), P2 (x2, y2, z2), P3 (x3, y3, z3) as characteristic points of the object define plane

\[ Ax + By + Cz + D = 0 \]  \hspace{1cm} (2.28)

The implicit constraint of four coplanar points would be: the fourth point P (x, y, z) is in the plane if the dot product of \( \overrightarrow{P_1P} \) with cross product of \( \overrightarrow{P_1P_2} \), and \( \overrightarrow{P_1P_3} \) equals zero. The following has to be satisfied:

\[
\begin{vmatrix}
    x-x_1 & y-y_1 & z-z_1 \\
    x_2-x_1 & y_2-y_1 & z_2-z_1 \\
    x_3-x_1 & y_3-y_1 & z_3-z_1
\end{vmatrix} = 0
\]  \hspace{1cm} (2.29)

The coefficients A, B, C and D in (2.28) are defined as the functions of the coordinates of three points, P1, P2 and P3. Only points with (x, y, z) satisfying the equation will be coplanar with these points.

Modifications of the geometry can be accomplished by modifications of the constraints due to repositioning. The new characteristic points solved for by the Newton-Raphson numerical method derive the new geometry. The input of precise geometry can be eliminated, which would very naturally be preferred by the designer in the early design stage. Also, the constraints can be other features than dimension, such as the area of the part shape, which depend on the user requirements and are more flexible. The new geometry will be displayed on the computer. The invalid constraint scheme must be corrected in the process.
Todd [1992] constructed geometry from a set of constraint values to a set of geometric positions. The position information is recorded as a set of linear equations according to the sequence of construction. The geometry can be solved for by using constructive geometry as a function of the constraints. The geometric information can be used in static analysis naturally. The derivative provides access to kinematic and dynamic analysis. However, the structure singularity needs to be identified interactively by the user.

For convenience, dimensions in the design should be treated as variables instead of fixed values, and they should be tolerated. Aldefeld [1988] used a rule-based geometric reasoning system to model the geometric structure.

Chung claimed that variational design could "provide constraint-driven capacity applied to a coupled combination of geometric constraints and engineering equations" and it is "a generic approach for dimension-driven designs." He also stated that parametric design "utilizes special case searching and solution techniques to provide dimension-driven capacity applied to primarily uncoupled geometric constraints and simple equations." He pointed out that tolerance analysis, mechanism analysis and design optimization could be performed on models developed with a variational design system [Chung et al. 1990].

The Monte Carlo method is a variation simulation method, which is used to predict variations in mechanical assemblies [Knappe 1963, Bohlin et al. 1970, Spotts 1980, Vocaturo 1983, Craig 1988, Lin 1987]. It uses random process sampling techniques to simulate assembly processes. Given the geometric standard and tolerance distribution of each component and using a random number generator, the process of selecting components is simulated. Nonlinear assembly functions need to be solved for the corresponding dependent variables. The sample of assembly resultant nominal is used to create a histogram for the statistical distribution after the numerous simulations. The mean and the departure from the mean can be estimated from the plot, which shows the mean and variation of the resultant in the assembly. This produces an accurate estimate of the variation. Variational Simulation Analysis (VSA) is one popular package. The variation of the output parameter as final production of the dimensions can be predicted statistically.

Variational Solid Modeling has also been used for tolerance analysis in the surface-based approach [Gupta et al. 1993]. Gupta mentioned two important applications benefit from a variational modeling capacity:

"1. In variational design systems, we can use a variational model to create nominal models of different sizes and shapes (large variations).
2. In solving tolerancing problem, we can use a variational model to represent different manufacturing instances (small variations).

Variational systems can be used for tasks which are not well understood or in investigation, such as in early stages of conceptional design for new products [CIME 1989].

2.5 ERROR SENSITIVITY ANALYSIS

Accuracy of mechanism output is affected by the manufacturing tolerance of each link's dimensions. The research into error analysis of mechanism through the sensitivity is very promising. The sensitivity of a mechanism is defined as the ratio of the change of a given output variable to the change in a design parameter [Faik et al. 1991]. The sensitivity of an assembly is defined as the derivative of the assembly variable with respect to the independent variables [Chase et al. 1989]. It is used as a measure of accuracy. Extensive studies have been done in this area.

Knappe defined the assembly sensitivity in the mechanical assembly as a partial derivative using explicit analytical expression. He showed the total variation as the summation of variation induced by each dimensional tolerance. He used Monte Carlo simulation to get the frequency distribution of output tolerance [Knappe 1963].

Sharfi presented a method for tolerance and clearance allocation as the post synthesis based on the tolerance sensitivities of linkage [Sharfi et al. 1983]. Will showed the process for tolerance assignment by the sensitivity value [Will 1993].

The change in configuration parameter of the four point, five point precision location problem has been analyzed from sensitivities [Mahableshwarkar et al. 1990].

Cleghorn identified the most sensitive link dimension and period within the entire cycle of a planar mechanism. He used an optimization method for the weighted allocation of the input dimensional tolerance band for a specified allowable limit of output. In later research, the mechanism is described as a combination of various link groups. The output of position from the former group was used as the input for the current group in a consecutive manner. The extensive error evaluations will be decreased by small groups. It simplified analysis [Cleghorn et al. 1988, Fenton et al. 1989].

Dimensionless entries have been used for a sensitivity analysis of a four-bar mechanism through transformation. The sensitivity to tolerance of the four-bar linkage
angles, in terms of the two nondimensional quantities, has been described. The trace of
these quantities is plotted in the four-bar solution space prescribed by link ratios [Faik et al.
1988, 1991]. The cycle profiles of the dimensionless position sensitivity for a balanced
and unbalanced crank-rocker have been plotted for the analysis through link ratios [Benner
et al. 1992]. It could be used to help the synthesis process.

Wu presented the Transmission Merit Parameter (TMP) for the transmission quality
and output sensitivity analysis of a mechanism. The effect of a dimensional disturbance is
researched. A constrained Jacobian matrix has been derived from differentiation of the
independent relationship among the coordinates. It is used to map the relationship between
the input and output variables including velocities, forces and variations. TMP is used to
generalize transmission index, which is the determinate of the derivative matrix of the
constraint equation with respect to the output variable [Wu et al. 1992]. Lee proposed the
sensitivity synthesis method for four-bar linkages for three and four positions. It would
allow the designer to synthesize a planar four-bar with the desired sensitivity value for three
positions if the relationship between the transmission angle and the performance sensitivity
is included [Lee et al. 1992].

Garrett statistically analyzed mobility bands of the linkage as a difference between
the ideal designed and the actual generated functions. The band varies around the designed
function due to the tolerance and clearance. The tolerance and clearance are generated by
the pseudo-random number and the mobility bands are evaluated by the Delta method, i.e.,
statistical 3σ method [Garrett et al. 1969]

Baumgarten used probability theory to analyze a four bar coupler's position error.
The variance of the coupler point and the tolerance band have been plotted [Baumgarten et
al. 1985].

2.6 LIMITATION OF CURRENT RESEARCH

Many methods have been reviewed in the previous section. Each has some
limitations when it is used in assembly tolerance analysis. The application to tolerance
analysis in the early design stages is particularly challenging due to uncertainty of the
design configuration and tolerances.

Type synthesis based on graph theory provides a systematic way to create unbiased
kinematic chains and mechanisms at the conceptional stage. A topological network graph
can highlight the connectivity. When used in tolerance analysis, it can help to generate the
loops in the modeling process. Also, it helps in counting the variation sources and kinematic variables.

Network graphs are highly abstract topological representations of mechanisms, for which dimensional and functional information are not provided. The separation of the topological structure from dimensions and functions of the mechanism limits attempts to apply this method to tolerance analysis. It has no kinematic characteristics until the types and dimensions of the mechanism are decided. By itself, it can not satisfy the functional requirements specified by the designer. Tolerance analysis is difficult to apply at this stage. Tolerance analysis is usually applied after all the design synthesis is done.

Traditional CAD/CAM packages supply methods for creating specific initial relationships with existing geometry and defining geometric dimensions precisely. However, the designer involved in the preliminary design is often not sure what configuration will satisfy all the requirements. So the rigidity imposed to specify the location and dimension does not encourage creativity or support the evaluation of design alternatives [Chung et al 1990]. Tolerance analysis can be done based on the model derived from CAD. However, the analysis results are represented as data corresponding to the specific design. Trial and error, or experience are needed for this design. Each alternative geometry requires regeneration of the CAD model and complete analysis. "What if" studies are not convenient if more nominal and tolerance changes are to be made. The possible extensive usage of the tolerance analysis result at early design stage therefore is limited.

Tolerances are assumed symmetric in order to simplify the analysis procedure. In manufacturing, variations often exhibit some process trends. Mean shift due to tool wear or environmental factors should be considered. For example, the diameter of a roller would increase as the wear of the cutting tool progresses. Not including directions of variations makes the design more conservative than it appears. The distribution of the variation is very difficult to guess. Even though a normal distribution is suggested in many analyses, the results may not be convincing due to this assumption.

Robust design would consider the variation in the design optimization. In mechanical assemblies, the independent variables are the manufacturing variables, the tolerances of which would be specified. Both the dimensional and the feasibility constraints are needed as closure constraints of the mechanical assembly imposed in the situations. This would cause some difficulty if the explicit expression of constraints is not
available. Also, tolerances of the dependent variables could only be decided through the
tolerance analysis of the assembly. They cannot be assigned at the same time as the
tolerances of independent variables.

Taguchi method is generally applied to unconstrained problems. The implicit
dimensional and feasibility constraints in mechanical assemblies are difficult to handle by
this method. The application of response surface method in tolerance analysis still has
more work to be done.

Variational geometry systems provide flexibility in design. They have been
demonstrated as a method for early design. However, most of them only deal with the
nominal dimensions at the design stage. Gupta mentioned that the procedure for solving
tolerance problems should be the same as the variational nominal, but with less variation
[Gupta et al. 1993]. Each instance of the assembly would need a complete variation
geometry model to describe and derive. Chung mentioned tolerance analysis as a future
implication. It was from data analysis approach, based on the created nominal dimensions
of the model, which is derived from variational system. One data analysis is required for
each model [Chung et al. 1990]. Not much attention has been paid to considering tolerance
analysis in parallel with a variational geometry system. A Monte Carlo method could
produce an accurate estimate, but the intensive simulation for the validity in statistical sense
would make its usage prohibitive in the early design stage.

The approaches for sensitivity analysis above are still very data-oriented and
localized. The transformation to dimensionless parameters makes analysis not directly
assembly tolerances, methods to reveal the relationship in a geometric way need to be
explored further.

The methods of system moments, the quadrature method, and the reliability index
are other statistical methods [Chase et al. 1991]. The statistical distribution of the assembly
function could be derived through these methods, but the results are still applied to a
specified design. The unavailability of explicit expressions for the dependent variables or
assembly functions limits the usage of these methods.

2.7 SUMMARY

From the above, we can see there is a great need for a relationship-oriented method
for qualitative and quantitative tolerance analysis. Early design will prefer the analysis to
be more flexible and effective since nominal changes are most likely to be made. There should be some way to preserve assembly relationships. Also, from the assembly tolerance consideration, the variation propagation mechanism needs to be better understood. The nature of tolerance sensitivity also needs to be better understood. More realistic analysis methods will lead to future possible decreases in cost and improvement in quality, if available. It will bring greater understanding of the effects tolerances on performance and quality. Decisions made upstream in the design stage based on the tolerance analysis will bring great benefit, as design changes at the early stage are much more economical than later, so that manufactured products could function satisfactorily and be produced economically.

This chapter reviewed several different methods for including variation in design and discussed their limitations. The great need for better tolerance analysis methods has been explained, which are more flexible and informative to be used at early design stage.
Chapter 3
SURFACE METHODS

Tolerance design is an iterative process. The numerous variation sources and process choices result in a vast array of possible combinations for the design variables. This challenges the traditional one-variable-at-a-time techniques.

The mathematical model of the sensitivity and the variation in assemblies involves loop equations, derivatives and matrix calculations, which make early design analysis tedious. Detailed analysis of tolerances in certain regions of design space requires extensive calculations. A simplified approach to tolerance design is presented in which each important design function is described as a surface in multiple-dimension space. Each axis represents a range of values of design parameter. By constructing surfaces for critical design variables, a design may be evaluated over a region of design space. Regions may be found over which variation is minimum. A flat section of the surface indicates a design neighborhood which is robust to manufacturing variation. The minimums and maximum can be obtained for the surfaces, which are of most concern to the designer.

Two approximate surface methods (variation response surface and quadratic variance) are proposed, which will provide an estimation of assembly tolerance over a region of design space. A third method is presented, which combines statistical tolerance analysis method with optimization techniques. The real tolerance surface is computed and maximum and minimum are found. It includes the solution of the system of nonlinear assembly equations for the nominal design variables, and the solution for assembly variations by the linearized method at each step. Implicit constraints are also enforced at each step. The real surface method is accurate, but requires more effort and cost. It was primarily for comparison with the first two approximated methods. It is made possible due to the availability of numerical methods and computing facilities.

In this chapter, each method will be discussed in detail. The three methods are compared through a set of tolerance design problems in the next chapter.

3.1 RESPONSE SURFACE METHODOLOGY

Response Surface Methodology

RSM, or Response Surface Methodology, refers to an existing statistical design and analysis tool. It can be used to predict the relationship between the response (output) of a
system and a number of input variables, e.g., factors. It fits the data from carefully designed experiments and maps the relationships within the experimental region. Through RSM, the designer can make an efficient empirical exploration of a system.

RSM has broad application which may be divided into two main areas: experimental design and data analysis techniques. It involves an experiment strategy, mathematical methods, and statistical inference. When applied to tolerance analysis, the input variables are design dimensions of the component parts of an assembly, which are subject to manufacturing process variation. The nominal values of dimensions are the design parameters which define design space. The design parameters of interest, such as the predicted nominal and variations of critical assembly variables, are the output or response variable generally. RSM presents tolerance analysis in a specified design space, whose limits are determined by physical engineering requirements.

Experimental Design

Once the proper range of the experiment is decided, experimental design procedure will decide what experimental "design points" are to be used. In statistical experimental design, a special group of select points is correctly chosen and the experimental run is done to investigate the response function relationship. The design considerations include the number of runs required, e.g., experimental effort, the level of the factor considered, the sequential buildup, the accuracy of the model, etc. It will ensure that the analysis will be as economical and as accurate as possible.

Quadratic models are frequently used in RSM because they adequately represent many scientific phenomena and are simple to work with. For the quadratic fitting, i.e., second order response surface, the test allows the coefficients in the quadratic model equation to be estimated. At least three levels of each variable are required for second order fitting. The simplest design is the $3^K$ Factorial Design, where K is the number of independent design parameters. For a factorial experiment, the effect of changing one variable can be assessed independently of the others. All possible combinations of three levels for each independent variable are run as experiments, resulting in $3^K$ experiments. For large K, it requires a large number of runs, so it is very wasteful.

Central Composite Design is a more efficient experimental design for fitting a second order model. It enables the estimation of all first order and quadratic terms, including two factor interaction terms. It is built upon two-level factorial design, and augmented by adding a set of points. The experiment can begin from the $2^K$ factorial
experiment. If this simple representation is not adequate, more points can be added to fit the second order response surface. This design gives a preferred alternative to the $3^K$ factorial design due to its requirement of fewer experiments and its versatility.

Coded Value and Design Matrix

In the representation of an experimental design, it is convenient to "code" the independent variables, with -1 representing the low level of the variables, +1 the high level and zero the center of the design. This corresponds to the transformation

$$X_c = \frac{X_a - X_m}{X_h - X_l}$$  \hspace{1cm} (3.3)

Here:
- $X_c$: the coded value for the experimental variables,
- $X_a$: the actual value in the original units,
- $X_m$: the middle value in the original units,
- $X_h$: the high value in the original units,
- $X_l$: the low value in the original units.

The half space between the low and high level on the variable $x$ is equivalent to one unit in coded variables. In terms of the coded variables, the experimental scheme may be seen clearly as a design matrix.

The following plots show the Central Composite Design for two and three factors and their design matrices:

![Central Composite Design for two factors](image)

**Figure 3.1** Central Composite Design for two factors
In Figure 3.2, the corner points, or vertices of the cube, are the high and low values for each factor in all possible combinations. The star points are the high and low points taken one at a time. Center point is located at the middle of the cube.

Number of Experiments

As stated earlier, the total number of experiments required to perform Factorial Design for three levels is $3^K$.

The number of total experimental points needed for Central Composite Design is

$$\text{Total} = \text{Factorial} + \text{Star} + \text{Center}$$  \hspace{1cm} (3.1)

In terms of number of the factors, for the $k$ factor, it becomes:

$$\text{Total} = 2^k + 2k + \text{Center}$$  \hspace{1cm} (3.2)

$2^k$ is the number for factorial design, corresponding to the corner points. They have high and low values for each factor.
2K is the number of star points. They are arranged along the axis of the variables with coded distance \( \alpha \), and symmetrically positioned with respect to the factorial frame. Various criteria can be used to choose \( \alpha \) for star points. If \( \alpha \) is other than one, there will be five levels for each factor in Central Composite Design.

The center point of the design only needs one experimental run. The center-point replications are not considered here, as the replications are only required for stochastic systems.

The numbers of tests required to run a \( 3^K \) Factorial and Central Composite Design for factors up to four are listed in the following table.

<table>
<thead>
<tr>
<th>Number of factors ( K )</th>
<th>No. coefficients in quadratic equation</th>
<th>No. runs required ( 3^K )</th>
<th>No. runs required CCD</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>6</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>27</td>
<td>15</td>
</tr>
<tr>
<td>4</td>
<td>15</td>
<td>81</td>
<td>25</td>
</tr>
</tbody>
</table>

It can be seen that the number of runs per coefficient estimated in Central Composite Design is in the efficient range of 1.5 to 2, while \( 3^K \) designs are seen very wasteful. Central Composite Design is clearly more efficient. For larger size problems, the cost of increasing the number of experimental runs due to the increment of the number of factors is the main prohibition for experimental design.

**Response Surface Model Fitting and Analysis of Variance**

Data obtained from experimental design is used to formulate a regression model. The quantitative and continuous response variables can be approximated over the interested region by a quadratic model.

The general quadratic model is

\[
\hat{Y} = b_0 + \sum_{i=1}^{k} b_i X_i + \sum_{i=1}^{k} b_{ii} X_i^2 + \sum_{i=1}^{k-1} \sum_{i<j=2}^{k} b_{ij} X_i X_j
\]  

(3.4)

Here:

\( \hat{Y} \): predicted response variables,
X: independent factors,

b: coefficients in the model, bij as coefficients of the cross product terms, bii as coefficients of the square terms.

The quadratic model is the best and simplest model to consider in the absence of the special knowledge about the system. It allows the prediction of slope and curvature for the response variables Y versus input variables X. It is also very flexible and can describe a wide variety of different surfaces. Because of the linear relationship in the regression coefficients, the model becomes a linear regression model. For the estimation procedure, the method of the general linear model still applies. To use the general linear multiple regression model in this case, X, X_i^2 and X_iX_j need to be redefined, and each regression coefficient is estimated from the data.

The quadratic equation is fitted to the experimental data through the least square technique. If the model is an adequate representation in the region of interest, the fitted surface will be an approximation of the actual system. In any assembly, the theoretical models are generally nonlinear and lack simplicity. The fitting spreads the error in regression over the whole region. The quality of regression needs to be analyzed.

Analysis of variance is used to check whether or not the chosen model is adequate to see if there is a significant regression relationship.

The total variation is subdivided into two parts, i.e., regression and error. The following notations will be used in the analysis.

n: total number of runs in the experiment,

k: number of coefficients in the model,

\( \bar{Y}_i \): predicted value by regression model,

Y_i: response value by experimental design,

\( \bar{Y} \): mean of response,

df_{reg.}: degrees of freedom of the regression model, df_{reg.} = k,

df_{err.}: degrees of freedom of the error, df_{err.} = n - k - 1,
\[ k+1 \] is the number of model coefficients, including the mean,

\[
SSR: \text{the sum of squares due to regression.} \quad SSR = \sum_{i=1}^{n} (\hat{Y}_i - \bar{Y})^2 \quad (3.5)
\]

It represents the variation of the predicted value about the mean of the response. It means the variability explained by the regression model.

\[
SSE, \text{the sum of squares due to error, represents the derivation of the response from its predicted value or fitted value.} \quad SSE = \sum_{i=1}^{n} (Y_i - \hat{Y}_i)^2 \quad (3.6)
\]

This includes the unexplained variation by the regression model, i.e., the variation of response about the regression model.

\[
SST: \text{the total sum of squares.} \quad SST = \sum_{i=1}^{n} (Y_i - \bar{Y})^2 \quad (3.7)
\]

This gives the variability among the response; it represents the deviation of the response about its mean.

\[
SST = SSR + SSE \quad (3.8)
\]

MSR: mean square-regression model, it is

\[
MSR = \frac{SSR}{df_{reg}} = \frac{\sum_{i=1}^{n} (\hat{Y}_i - \bar{Y})^2}{k} \quad (3.9)
\]

MSE: mean square-error.

\[
MSE = \frac{SSE}{df_{err}} = \frac{\sum_{i=1}^{n} (Y_i - \hat{Y}_i)^2}{n - k - 1} \quad (3.10)
\]
Because there is no "pure error" (i.e., measurement error) involved in the calculation of tolerance, MSE is more appropriately called the "mean square lack-of-fit."

\[ s = \sqrt{\text{MSE}} \]  
\[ (3.11) \]

The "s" is an unbiased estimated standard derivation about regression.

\[ R^2 \] is the coefficient of multiple determination. It illustrates the adequacy of a fitted regression model. It represents the variability that is explained by the regression model as the fraction of the total sum of squares.

\[ R^2 = \frac{\text{SSR}}{\text{SST}} = 1 - \frac{\text{SSE}}{\text{SST}} \]  
\[ (3.12) \]

\[ R^2 \] is in the range: \( 0 \leq R^2 \leq 1 \).

\( R^2 \) provides summary statistics to show that how well the regression equation fits the data. The regression model reduces the variability in the value of \( y \) by \( R^2 \). A higher \( R^2 \) indicates that the regression model explains a large proportion of the variability among the observations. If it equals exactly one, it means that all points fall on the proposed regression model, i.e., perfect fit case.

If \( R^2 \) is adjusted by degree of freedom,

\[ R_{\text{adj}}^2 = 1 - \frac{\text{SSE/df}_{\text{err}}}{\text{SST/df}_{\text{tot}}} \]  
\[ (3.13) \]

\( R_{\text{adj}}^2 \) is the unbiased estimation of the population of \( R^2 \).

Overall F statistic is

\[ F_{\text{df}_{\text{reg}}, \text{df}_{\text{err}}} = \frac{\text{MSR}}{\text{MSE}} = \frac{\text{SSR}}{\text{SSE}} \cdot \frac{\text{df}_{\text{err}}}{\text{df}_{\text{reg}}} \]  
\[ (3.14) \]

It is used to test whether all of the regression coefficients could be zeros. If it is larger than the tabulated critical F value, it indicates that not all of the coefficients are zeros.

As

\[ F = \frac{R^2}{1 - R^2} \cdot \frac{\text{df}_{\text{err}}}{\text{df}_{\text{reg}}} = \frac{R^2}{1 - R^2} \cdot \frac{n - k - 1}{k} \]  
\[ (3.15) \]

Then a higher \( R^2 \) implies a large value for the F-ratio.
The significance of each individual regression coefficient can also be assessed [Hogg et al. 1994].

The partitioning of the total sum of square for Multiple Linear Regression Model is summarized in the following table:

<table>
<thead>
<tr>
<th>Source</th>
<th>degrees of freedom df</th>
<th>Sum of Square SS</th>
<th>Mean square MS SS/df</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regression</td>
<td>k</td>
<td>SSR</td>
<td>MSR=SSR/k</td>
<td>MSR/MSE</td>
</tr>
<tr>
<td>Error</td>
<td>n-k-1</td>
<td>SSE</td>
<td>MSE=SSE/(n-k-1)</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>n-1</td>
<td>SST</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Analysis of Fitted Surface**

Once the analysis of variance indicates the adequacy of the fitted surface, the response surface can be explored. The optimum operating conditions can be determined by the analysis of the fitted surface. It can provide valuable information over the region for the designer, such as how the response is affected by the input variables, where the maximum or minimum response is, etc. As an approximate mapping of a surface within a limited region, a contour plot of the surface of interest is useful. Stationary point analysis may provide a choice of the desired design dimension values.

For the quadratic model, it can be represented in matrix form.

\[
Y = b_0 + b^T X + \frac{1}{2} X^T H X
\]  

(3.16)

\(b_0\): constant,

\(b\): vector of coefficient of linear terms,

\(X\): vector of variables, or coded independent variable levels,

\(H\): Hessian matrix, symmetric matrix with the twice coefficient of the square terms on the diagonal and coefficient of the cross product terms on the off diagonal.

After differentiating with respect to each independent variable and setting the result zero, the stationary point of the quadratic equation can be expressed as:
\[ X_0 = -H^{-1}b \]  

(3.17)

... There are basically three kinds of stationary points: maximum, minimum and saddle. They can be decided through second order derivatives.

The nature of stationary points also can be found by examining the eigenvalues of the Hessian of Equation 3.16. Eigenvalues are determined by solving the characteristic equation

\[ |H - \lambda I| = 0 \]  

(3.18)

If all eigenvalues \( \lambda \) are positive, this stationary point represents the minimum of the fitted surface. Moving away from the stationary point, the value of fitted surface will increase. If all eigenvalues \( \lambda \) are negative, the stationary point represents the maximum of the fitted surface. If the eigenvalues differ in sign, the stationary point is a saddle point, where the experiment may get an increment or decrement when moving away from the point.

Also, the following equation of quadratic form can be obtained by rotation and transformation of Equation 3.16. Here, \( A \) is symmetric,

\[ f(x') = x'^{T}Ax' \]  

(3.19)

If \( A \) is nonsingular, the new origin will be the only stationary point. The function has either a global minimum, maximum, or saddle point at the origin point. They will be decided by the definite condition of \( A \). If the determinant of each principle minor matrices are positive, \( A \) is positive definite and a minimum of the function occurs at the origin point. If \(-A\) is positive definite, the \( A \) is negative definite, and the function has a maximum value at the origin point.

If the transformation makes the expression have only "square terms,"

\[ g(w) = w^{T}Bw \]  

(3.20)

i.e.,

\[ g(w) = \lambda_1 w_1^2 + \lambda_2 w_2^2 + ... + \lambda_n w_n^2 \]  

(3.21)

then, \( \lambda_i \) are eigenvalues. The sign and magnitude of eigenvalues provide more geometric information about the system. For example, if \( k=2 \), \( \lambda_1 < 0 \), \( \lambda_2 > 0 \), the
stationary point represents a saddle case. Moving along the \( w_1 \) axis away from the stationary point in either direction, a corresponding decrease in estimated response will be obtained. Along the \( w_2 \) axis, increase will happen. It implies the existence of a system containing two peaks, where two maximums occur in different regions. The magnitude of the eigenvalue represents the elongation of the contours. Figure 3.3 shows one example. The contours are elongated along the axis corresponding to the smaller eigenvalue in the magnitude [Myers 1971].

![Eigenvector Diagram](image)

Figure 3.3 Maximum elongated along \( w_1 \) axis

If one of the eigenvalues is close to zero, a near-stationary-ridge will happen, where the possible operating condition will be a range rather than a point. The contours of the function will appear to be nearly parallel lines.

If there are more independent variables, the surface may be interpreted mathematically without contour plots. For constrained problems, it may necessary is to find the value of design dimensions which produce a maximum or minimum response if the stationary point is out of the boundary of feasible design space. Other mathematical tools such as constrained optimization accommodated with numerical methods can be used.

**Application to Tolerance Analysis**

When applied to tolerance analysis, it can be called the Variation Response Surface method, i.e., VRS. It follows these steps:

1. Determine the response variables, which are usually the dependent assembly parameters critical to design performance.

2. Determine the component dimensions and specified tolerance of the contributing independent variables.
3. Assign realistic ranges for the acceptable nominal values of the important independent design parameters.

4. 'Run' the experiment, in other words, do the tolerance analysis at each assigned experimental design point. This involves solving the nonlinear system of equations and using linearized method for the nominal values and statistical variations of the dependent assembly variables.

5. Fit a quadratic model to the experimental results over the range defined. The model will be used for estimating the response function without repeating the solution procedure at each point in design space. Quadratic models may be fit to the nominal assembly variables, its sensitivity to any single design variable, or to the variation results.

6. Evaluate the adequacy of a model by analysis of variance technique.

7. "View" the estimated surfaces. This is an interactive process. If the designer is not satisfied with the accuracy or result, the range of the design can be shifted according to the approximation or more points may be added for a more accurate fitting.

8. Use contour, stationary point, or other techniques such as constrained optimization on the fitted model for further analysis.

### 3.2 QUADRATIC VARIANCE

In the response surface method, a quadratic surface is fit to a set of experimental points. As each experimental point, the nominal, sensitivities or the variations of the desired assembly variables may be calculated from the system of assembly functions. A quadratic surface may be fit to any of these sets of desired values. Thus, we could obtain fitted surfaces for the nominal, the variation or any of the sensitivities.

An alternative way fits only the nominal value of a critical assembly variable with a quadratic model. Then the sensitivities and variance are determined by taking derivatives of the derived equation for the nominal quadratic surface. This method will be called the quadratic variance method (QV). It is attractive because it only requires one fitted surface, from which all other quantities of interest are derived.

Let \( Y(X) \) be a quadratic function of \( m \) independent variables. Then the variance function \( V_Y \) can be defined as:
\[ V_Y = \sum_{i=1}^{m} \left( \frac{\partial Y}{\partial x_i} \right)^2 \sigma_{x_i}^2 \]  \hspace{1cm} (3.22)

Where \( \sigma_{x_i} \) represents the standard derivation of the independent system variables. Equation 3.22 is a variance transmission formula. It reflects the change effect of the independent variable to the dependent variance. Usually, minimizing variance caused by fluctuation produces robust design. The minimum variance occurs at the stationary point of the function. This can be seen clearly from the equation. As the gradient of function is zero at the stationary point, it will produce zero variance. Other than the stationary point, the variance will always be positive due to the square effect.

For non-quadratic functions, based on Rolle's theorem, the maximum variance will occur somewhere between two stationary points. This will happen if the variance function is continuous on the closed interval and differentiable on the open interval, where the intervals are produced by these stationary points.

The variance function by Equation 3.23 is always positive definite for quadratic functions, since the Hessian in the following equation is positive definite [Webb et al. 1995].

\[ V_Y = V_Y^0 + (\nabla V_Y)^T(\Delta X) + \frac{1}{2}(\Delta X)^T(\nabla^2 V_Y)(\Delta X) \]  \hspace{1cm} (3.23)

\[ \nabla V_Y = 2H\Sigma^2 \nabla Y \]  \hspace{1cm} (3.24)

\[ \nabla^2 V_Y = 2H\Sigma^2 H \]  \hspace{1cm} (3.25)

Here:

\( \nabla V_Y \): gradient of the variance,
\n\( \nabla^2 V_Y \): Hessian of the variance of a quadratic function,
\n\( H \): Hessian of quadratic function \( Y(x) \), which is a constant matrix,
\n\( \Sigma^2 \): the diagonal matrix with \( \sigma_{x_i}^2 \) as the elements,
\n\( \nabla Y \): first derivative of the quadratic function \( Y(x) \).
The quadratic function $Y(X)$ describes a surface in $m$-dimensional space. The variance surface is derived from the quadratic surface. There is always minimum variance regardless if the original function is positive definite, negative definite or indefinite [Webb 1994]. The definite conditions of the Hessian matrix and the extremes represented by stationary points for an original quadratic function and its variance function are listed in the following table. The shapes of surface are visible for three dimensional space and they are listed also.

<table>
<thead>
<tr>
<th>Hessian</th>
<th>positive definite</th>
<th>negative definite</th>
<th>indefinite</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shape of surface</td>
<td>convex</td>
<td>concave</td>
<td>saddle</td>
</tr>
<tr>
<td>Stationary point</td>
<td>minimum of function</td>
<td>maximum of function</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.4 Nature of variance function of quadratic function

<table>
<thead>
<tr>
<th>Hessian</th>
<th>always positive definite</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shape of surface</td>
<td>always convex</td>
</tr>
<tr>
<td>Stationary point</td>
<td>always minimum of variance</td>
</tr>
</tbody>
</table>

The minimum variance occurs at the minimum function point for the positive definite function, at the maximum function point for a negative definite function, and at the saddle point for an indefinite case. This conclusion can be easily drawn for unconstrained problems. For constrained cases, if the stationary point is outside of the boundary, optimization techniques are needed to find the minimum variance of the function in the bounded region. There is information about the variance in the original function.

The variance function has connection with the tolerance analysis. If the dependent variable can be expressed as an explicit function $Y(x)$, the variance function may be determined directly from Equation 3.22 by taking derivatives of $Y(x)$. For tolerance analysis, the dependent variables for a mechanical assembly are normally implicitly expressed in a nonlinear system of equations. By fitting a quadratic model to the nominal variables, the relationship to the independent variables is converted to an explicit function. If the model is adequate, derivatives may be taken with respect to the independent variables. The quadratic variance method finds the variation directly from the quadratic regression model. However, the nonlinear system of equations must be solved at a sufficient number of points to fit the quadratic nominal surface.
3.3 REAL SURFACE

The two methods discussed above can provide an estimate of the response in the range of interest. Due to the availability of the general package of optimization OPTDESX and its interface, a third surface method is proposed. This method operates on the real statistical tolerance analysis surface. It can be used as an individual surface method or as the reference to evaluate the accuracy of other methods.

OPTDESX

OptdesX is a window-driven software package implemented for general design optimization. It has been developed at BYU for several years [Parkinson 1992]. It provides the user a general system to set up and run optimization problems. The objective and the explicit constraint functions can be defined from design functions. The minimum or maximum of the design objectives can be found through a set of optimization algorithms. It is equipped with several general purpose nonlinear programming algorithms such as GRG (Generalized Reduced Gradient), SQP (Sequential Quadratic Programming), BB (Branch and Bound), etc. They can be used for continuous and discrete problems. The design variables and design functions can be conveniently mapped from the analysis variables and analysis functions; mapping can be simple or sophisticated. OPTDESX also provides ways to graph data created in the process.

OPTDESX requires problem-specific analysis routines, which the user must provide. These routines are connected with OptdesX by the driven routine. Therefore, it can be used to solve a wide variety of design problems.

Difficulties with Tolerance Analysis in Optimization

Statistical tolerance analysis can provide estimated variation results as discussed in Chapter 2. As the behavior of the variation in the design space needs to be analyzed, there arise some special difficulties compared with general optimization. There are two levels of problems that need to be considered: first, the value of the feasible nominal throughout design space, and second, the estimated variation at any specified nominal. In design space, any change of the independent variables X will change the dependent variables U. Enforced constraints in the assembly can not be explicitly defined. It requires an iterative solution of an implicit set of nonlinear constraint equations for U. The implicit constraints must be handled inside the user-provided program. The vector model for the assembly must be constructed and properly handled. Nominal dependent variables need to be
calculated, and the linearized method is needed to find the statistical tolerance analysis results for each step in design space.

**Opt-Tol Procedure**

Opt-Tol is a new way to link optimization with tolerance analysis. The integration of OptdesX and tolerance analysis provides a tool for the designer to find the optimal tolerance design. It can be used to design an assembly system which is robust to manufacturing noise.

The assembly vector loops are described by their relative position relationships in terms of chains of scalar lengths and angles. All independent and dependent variables must be identified and corresponding independent tolerances need to be assigned.

The assembly model is entered through ANAPRE. It is for pre-processing and provided by the user. It is used to initialize a complex model. ANAPRE is only called once when OptdesX is first started. In Opt-Tol, it reads in the template data file for the assembly. After this step, the model is defined, including all vector loops, variable names, nominal dimensions, independent or dependent variables, tolerances, successive orders, etc. The information is essential for building complete geometry of the assembly.

In setup, the design variables will be chosen from several analysis variables. The design space is assigned as within the minimum and maximum value of the design variables. Objective functions and constraints are defined from design functions. In assembly analysis, design variables are independent variables; the objective function can be a variation function of the dependent variables or other user-designed function.

ANAFUN is also user-provided. During each analysis call, OptdesX adjusts the nominal of design variables, and return them to ANAFUN. In ANAFUN, the current vector loops are constructed, and used to generate algebraic equations which relate independent variables, i.e., manufacturing component dimensions, to dependent variables, i.e., assembly dimensions. Each evaluation includes iterative nonlinear nominal solutions and the linearized variation evaluations. In Opt-Tol, ANAFUN calls a subroutine to find the closure solution of the nonlinear system of equations for the dependent nominal values through an iterative process. The implicit closure constraint has to be satisfied for the assembled case. Then, all the tolerance sensitivities are obtained. The dependent variations, i.e., result assembly variations are evaluated in terms of the tolerance of the independent variables and their sensitivities using statistical model for tolerance
accumulation. The new values of design function are passed back to OptdesX. The
returned function value will be used to calculate the gradient information, which decides the
directions for subsequent search.

![Diagram](attachment:image.png)

Figure 3.4 Information flow between OPTDESX and tolerance analyzer

Opt-Tol provides automatic movement between the modeling and optimization
environment once the model is initialized. It eliminates repeating assembly modeling in the
design space search. These interactive processes are placed inside of a computerized
optimization loop. It will find the optimal tolerance design if the optimal criteria become
satisfied. Normally, the designer is most concerned about the minimum or maximum
variation point. The connection of general optimization package OPTDESX and tolerance
analyzer makes this true function analysis possible.

3.4 SUMMARY

Three surface methods have been introduced for tolerance analysis. VRS provides
the estimation of the variation from regression of the experimental data. Quadratic variance
gives the variation through the derivative of the nominal quadratic estimation. Opt-Tol
gives the real tolerance surface, which finds statistical variation analysis at each step and
finds the maximum and minimum variation through the connection with the general
optimization package OptdesX. Case studies for the surface methods will be presented in
the following chapter.
Chapter 4
SURFACE METHODS - CASES STUDIES

This chapter presents case studies using three surface methods: Variation Response Surface Method (VRS), Quadratic Variance method (QV), and Real surface method (Opt-Tol).

The vector assembly model is used in each method. A nonlinear system of equations can be obtained from the vector loops. They implicitly express the relationship between the manufactured and the assembly dimensions. They are used to solve for the nominal values of the dependent variables through an iterative process. The linearized method discussed in Chapter 2 is used afterwards. It finds the tolerance sensitivities and the dependent variations in terms of the independent tolerances. The sensitivities are evaluated at the nominal dimensions, which include all independent and dependent dimensions.

For the Variation Response Surface, the procedure is performed for the experimental points and the nominal and variation surfaces are fitted. Further analysis is performed based on the estimated surfaces. The Quadratic Variance method uses only the nominal fitted surface from VRS and then takes the derivative to obtain the variance function. The square root of the variance gives the estimate of the variation in the assembly, as discussed in Chapter 3. The Real surface method uses user interface to enter the vector model. It also uses an iterative process to find the kinematic nominals, takes the derivatives numerically, and calculates the dependent variations. OptdesX guides automatic movement between the optimization and the analysis model by which the dimensions of the design variables are updated and objectives are obtained and returned. The maximum and minimum of the surface in the design space can be obtained. It is the most accurate method and is used as a standard reference for comparison. By all three methods, the surfaces and the variations are found, as well as the maximum and minimum value. The summary and comparison are in the last section.

4.1 TAPEHUB EXAMPLE

Problem Description

A tapehub is a locking mechanism that is used to mount and hold a magnetic tape reel in place on a drive hub. By symmetry in geometry, the assembly can be reduced to a two-dimensional model to complete the analysis.
Figure 4.1 shows the assembly vector model, which is overlaid on the radial cross section of the mechanical assembly. The plunger can slide vertically along the guide, and the arm slides horizontally upon the base. If the plunger is pushed down by a spring, the wedge angle $\theta$ on its outer edge forces the arm and the rubber pad to slide outward until they rest against the tape reel. An interference fit compresses the rubber pad, secures the tape reel in place and allows the hub to rotate the tape reel in both directions. The figure shows the extreme position, when the tape reel is locked. Here, the clearance $\Delta x$, shown between the pad and the reel, has a negative value because the interference is in a press fit.

![Figure 4.1 Tapehub mechanism and its vector loops](image)

Two loops are used to model the assembly. Loop 1 is a closed loop (zero gap), and Loop 2 is a 1-D open loop (nonzero gap). There are nine manufactured dimensions as independent variables. All the variable names, descriptions, basic sizes and the assigned tolerances for manufactured variables are listed in Table 4.1. Four dependent variables can be determined from the assembly model. Their names, descriptions, design sizes are listed in Table 4.2. The last column shows the calculated assembly variations for a specified set of nominal part dimensions, that is, a single point in the design space, which were calculated statistically for the nominal dimensions shown by the linearized method as discussed in Chapter 2.

Two variations of the dependent variables are of interest in the assembly. One is the variation of resultant radius of tapehub $r_l$ in the locked position, which is a kinematic
variable in the closed loop. Another is the variation of the gap \( \Delta x \) between the arm and the reel, which is the clearance in the open loop. They are called \( d_{r1} \) and \( d_{\Delta x} \) respectively.

### Table 4.1 Manufactured dimensions for tapehub

<table>
<thead>
<tr>
<th>Variable Name</th>
<th>Description</th>
<th>Basic Size</th>
<th>Tolerance (±)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b )</td>
<td>Height of high step</td>
<td>0.390 in</td>
<td>.003 in</td>
</tr>
<tr>
<td>( a )</td>
<td>Location of plunger at bottom</td>
<td>1.360 in</td>
<td>.004 in</td>
</tr>
<tr>
<td>( \theta )</td>
<td>Wedge angle</td>
<td>75.00°</td>
<td>( 2^\circ )</td>
</tr>
<tr>
<td>( r )</td>
<td>Corner radius of arm</td>
<td>0.060 in</td>
<td>.002 in</td>
</tr>
<tr>
<td>( e )</td>
<td>Width of arm from center of corner</td>
<td>0.235 in</td>
<td>.003 in</td>
</tr>
<tr>
<td>( i )</td>
<td>Thickness of rubber pad</td>
<td>0.050 in</td>
<td>.002 in</td>
</tr>
<tr>
<td>( g )</td>
<td>Height of arm from center of corner</td>
<td>0.5275 in</td>
<td>.004 in</td>
</tr>
<tr>
<td>( h )</td>
<td>Height of low step</td>
<td>0.200 in</td>
<td>.003 in</td>
</tr>
<tr>
<td>( rt )</td>
<td>Radius of tape reel</td>
<td>1.780 in</td>
<td>.005 in</td>
</tr>
</tbody>
</table>

### Table 4.2 Assembly dimensions for tapehub

<table>
<thead>
<tr>
<th>Variable Name</th>
<th>Description</th>
<th>Basic Size</th>
<th>Variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u )</td>
<td>Dependent length</td>
<td>.365483 in</td>
<td>0.008193</td>
</tr>
<tr>
<td>( r_l )</td>
<td>Hub Radius in locked position</td>
<td>1.797549 in</td>
<td>0.014497</td>
</tr>
<tr>
<td>( \phi )</td>
<td>Dependent angle</td>
<td>15.0000°</td>
<td>( 2^\circ )</td>
</tr>
<tr>
<td>( \Delta X )</td>
<td>Interference between arm &amp; reel</td>
<td>-0.017549 in</td>
<td>0.015335</td>
</tr>
</tbody>
</table>

### Results for Tapehub from Design Space

In this section, the assembly variations are examined over a range of nominal values, which can be provided to the designers to help them to make decision. The design space is spanned by design variables \( b \), height of high step, and \( \theta \), the wedge angle of the plunger. More than two design parameters could be chosen, but two are sufficient to illustrate the methods.

### Table 4.3 Design space for tapehub

<table>
<thead>
<tr>
<th>Variable Name</th>
<th>Basic Size</th>
<th>Design Space (±)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b )</td>
<td>0.39</td>
<td>0.08</td>
</tr>
<tr>
<td>( \theta )</td>
<td>75°</td>
<td>10°</td>
</tr>
</tbody>
</table>
For each evaluated point, a valid set of dimensions must be found, which requires a solution of the nonlinear loop equations to determine the nominal values of the adjustably or kinematic, assembly variables. This is achieved by iterating kinematics variables by Newton’s method until loop closure is satisfied. The variation drl, for closed loop, and dΔx, for open loop in the design space were analyzed by all three surface methods. In VRS, the statistical variations, drl and dΔx, are calculated at nine experimental points and fitted as a quadratic surface. The estimated surface is used to search for the maximum and minimum variations. The fitting accuracy can be seen from Table 4.4. The coefficients of multiple determination, R², show corresponding percentage of the variability of fitted values, i.e., rl, drl, Δx or dΔx, can be explained by explanatory variables, i.e., b and θ. The s is unbiased estimated standard derivation about the regression. The fits are good for these four parameters.

Table 4.4 Fitting parameters for tapehub (VRS method)

<table>
<thead>
<tr>
<th></th>
<th>rl (nominal)</th>
<th>drl</th>
<th>Δx (nominal)</th>
<th>dΔx</th>
</tr>
</thead>
<tbody>
<tr>
<td>s</td>
<td>0.0009354</td>
<td>0.0001169</td>
<td>0.0009532</td>
<td>0.0001218</td>
</tr>
<tr>
<td>R²</td>
<td>100%</td>
<td>100%</td>
<td>100%</td>
<td>99.9%</td>
</tr>
<tr>
<td>R(adj)²</td>
<td>100%</td>
<td>99.9%</td>
<td>100%</td>
<td>99.8%</td>
</tr>
</tbody>
</table>

Table 4.5 Result and comparison of variation dlrl from three methods

<table>
<thead>
<tr>
<th></th>
<th>VRS minimum</th>
<th>QV minimum</th>
<th>Real minimum</th>
<th>VRS maximum</th>
<th>QV maximum</th>
<th>Real maximum</th>
<th>Base</th>
</tr>
</thead>
<tbody>
<tr>
<td>drl</td>
<td>0.01092655</td>
<td>0.00877053</td>
<td>0.010895</td>
<td>0.01987001</td>
<td>0.01822851</td>
<td>0.019886</td>
<td>0.01449</td>
</tr>
<tr>
<td>rl (nom.)</td>
<td>1.727758</td>
<td>1.727758</td>
<td>1.727758</td>
<td>1.905886</td>
<td>1.905886</td>
<td>1.905886</td>
<td>1.797549</td>
</tr>
<tr>
<td>b</td>
<td>0.47</td>
<td>0.47</td>
<td>0.47</td>
<td>0.31</td>
<td>0.31</td>
<td>0.31</td>
<td>0.39</td>
</tr>
<tr>
<td>angle</td>
<td>85°</td>
<td>85°</td>
<td>85°</td>
<td>65°</td>
<td>65°</td>
<td>65°</td>
<td>75°</td>
</tr>
</tbody>
</table>

Table 4.6 Result and comparison of variation dΔx from three methods

<table>
<thead>
<tr>
<th></th>
<th>VRS minimum</th>
<th>QV minimum</th>
<th>Real minimum</th>
<th>VRS maximum</th>
<th>QV maximum</th>
<th>Real maximum</th>
<th>Base</th>
</tr>
</thead>
<tbody>
<tr>
<td>dΔx</td>
<td>0.01202062</td>
<td>0.00877057</td>
<td>0.011987</td>
<td>0.02204880</td>
<td>0.01822851</td>
<td>0.020505</td>
<td>0.015335</td>
</tr>
<tr>
<td>Δx (nom.)</td>
<td>0.0522425</td>
<td>0.0522425</td>
<td>0.0522425</td>
<td>-0.125886</td>
<td>-0.125886</td>
<td>-0.125886</td>
<td>-0.017549</td>
</tr>
<tr>
<td>b</td>
<td>0.47</td>
<td>0.47</td>
<td>0.47</td>
<td>0.31</td>
<td>0.31</td>
<td>0.31</td>
<td>0.39</td>
</tr>
<tr>
<td>angle</td>
<td>85°</td>
<td>85°</td>
<td>85°</td>
<td>65°</td>
<td>65°</td>
<td>65°</td>
<td>75°</td>
</tr>
</tbody>
</table>
The maximum and minimum variations for \( d_{rl} \) and \( d_{\Delta x} \) are listed in Tables 4.4 through 4.6. The results from VRS and the QV methods are compared to the more exact or real variation values. The corresponding values for nominal dimensions are included as base for reference.

It can be seen that the three surface methods predict the same nominal design dimensions at which the maximum and minimum of the variations occur for both \( d_{rl} \) in closed loop and \( d_{\Delta x} \) in open loop. However, the values of the maximum and minimum variations from the three methods do not agree. The estimated variation values, \( d_{rl} \) and \( d_{\Delta x} \), from VRS are very close to Real. The \( d_{rl} \) has a minimum value which is 1/4 less variation than at original nominal dimensions, and \( d_{\Delta x} \) is 1/5 less. It looks as if the design may be improved in terms of the variation. However, at the location where minimum variation occurs, the nominal interference is positive, indicating a clearance. Therefore, other independent nominal dimensions need to be adjusted to compensate. In addition to the variation, the designer needs to consider the performance requirements in making design decisions.

Figure 4.2 Predicted dimensions for tapehub

The variations at the predicted dimensions are plotted as bar charts for comparison, which are included in the appendix in Figure A4.1. The accuracy of the QV method is affected by the number of variables in the design space. The discussion is included in the appendix.
4.2 REMOTE POSITIONING MECHANISM EXAMPLE

Problem Description

![Diagram of remote positioning mechanism]

Figure 4.3 Remote positioning mechanism and its vector loops

The Figure 4.3 above shows a remote positioning mechanism and its vector loops. The remote positioning mechanism consists of five parts plus the ground, which are connected with seven pin joints. It forms two parallelograms, which are described by closed loops loop 1 and loop 2. Link AB is the input link and can rotate around the pivot, which is located at the origin of the coordinates. Input angle $\theta_1$ is called as the reference angle. The purpose of the mechanism is to hold the position of the point P stationary for
any orientation of the input link. The orientation of parts 5 must also maintain parallel to part 1. However, manufacturing variations will accumulate in the assembly, which makes the position and the orientation of the P vary.

There are eight manufactured lengths from a to j. They are independent variables with specified tolerances determined by a manufacturing process. For parts 2 and 3, each has three joints and requires manufactured angle $\theta_3$ and $\theta_4$ to locate the third joint relative to the other two. The angles $\theta_2$ and $\theta_5$ are also manufactured angles. The other angles, $\phi_1$ through $\phi_6$, are all kinematic angles, thus they are dependent.

Two closed vector loops and one open loop, are used to model the remote positioning mechanism. There are six kinematic rotational angle variables in the closed loops, corresponding to the rotational degree of freedom of each pin joint, not including the input joint. The nominal angles can be solved simultaneously by solving the system of nonlinear equations defined by the closed vector loops.

The position and the orientation of the point P, as the open loop nominal clearance parameters, can be obtained by solving the open loop equations. All thirteen manufactured dimensions as well as their manufacturing tolerances are listed in Table 4.7.

Table 4.7 Manufactured dimensions for remote positioning mechanism

<table>
<thead>
<tr>
<th>Variable Name</th>
<th>Description</th>
<th>Basic Size</th>
<th>Tolerance $(\pm)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>Length of input bar</td>
<td>22.000 in</td>
<td>.005 in</td>
</tr>
<tr>
<td>b</td>
<td>Length of BC</td>
<td>10.400 in</td>
<td>.005 in</td>
</tr>
<tr>
<td>c</td>
<td>Length of CD</td>
<td>22.000 in</td>
<td>.005 in</td>
</tr>
<tr>
<td>d</td>
<td>Length between grounds</td>
<td>10.400 in</td>
<td>.005 in</td>
</tr>
<tr>
<td>$\theta_2$</td>
<td>Closing angle</td>
<td>59.75°</td>
<td>.5°</td>
</tr>
<tr>
<td>e</td>
<td>Length of CE</td>
<td>49.300 in</td>
<td>.005 in</td>
</tr>
<tr>
<td>$\theta_3$</td>
<td>Transferring angle</td>
<td>30.25°</td>
<td>.5°</td>
</tr>
<tr>
<td>f</td>
<td>Length of EF</td>
<td>12.900 in</td>
<td>.005 in</td>
</tr>
<tr>
<td>g</td>
<td>Length of FG</td>
<td>49.300 in</td>
<td>.005 in</td>
</tr>
<tr>
<td>i</td>
<td>Length of GC</td>
<td>12.900 in</td>
<td>.005 in</td>
</tr>
<tr>
<td>$\theta_4$</td>
<td>Transferring angle</td>
<td>42.646°</td>
<td>.5°</td>
</tr>
<tr>
<td>j</td>
<td>Length of arm</td>
<td>22.000 in</td>
<td>.005 in</td>
</tr>
<tr>
<td>$\theta_5$</td>
<td>Transferring angle</td>
<td>137.354°</td>
<td>.5°</td>
</tr>
</tbody>
</table>
The assembly nominal values, i.e., the dependent nominal values, and the variations are calculated and listed in Table 4.8. The variation of the point P, in both position and orientation, can be obtained through the linearized method if the manufactured dimensions and the tolerances are given.

Table 4.8 Assembly dimensions for remote positioning mechanism

<table>
<thead>
<tr>
<th>Variable Name</th>
<th>Description</th>
<th>Basic Size</th>
<th>Variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_1$</td>
<td>Dependent angle</td>
<td>59.749997°</td>
<td>0.502543°</td>
</tr>
<tr>
<td>$\phi_2$</td>
<td>Dependent angle</td>
<td>120.250006°</td>
<td>0.501181°</td>
</tr>
<tr>
<td>$\phi_3$</td>
<td>Dependent angle</td>
<td>59.749997°</td>
<td>0.500569°</td>
</tr>
<tr>
<td>$\phi_4$</td>
<td>Dependent angle</td>
<td>47.3540043°</td>
<td>0.868241°</td>
</tr>
<tr>
<td>$\phi_5$</td>
<td>Dependent angle</td>
<td>132.646°</td>
<td>0.867601°</td>
</tr>
<tr>
<td>$\phi_6$</td>
<td>Dependent angle</td>
<td>47.3540043°</td>
<td>0.866813°</td>
</tr>
<tr>
<td>$\Delta X$</td>
<td>X-cord of point P</td>
<td>-5.239254 in</td>
<td>0.667891</td>
</tr>
<tr>
<td>$\Delta Y$</td>
<td>Y-cord of point P</td>
<td>-58.283890 in</td>
<td>0.276332</td>
</tr>
<tr>
<td>$\Delta \phi$</td>
<td>Orientation of point P</td>
<td>0.000000°</td>
<td>0.709387°</td>
</tr>
</tbody>
</table>

Results for Remote Positioning Mechanism from Design Space

The design space for the remote positioning mechanism is shown in Table 4.9 and spanned by three design variables, b, e and $\theta_4$. Both b and e are shared by different loops. $\theta_4$, as the connecting joint angle, adjusts the relationship among the loops.

Table 4.9 Design space for remote positioning mechanism

<table>
<thead>
<tr>
<th>Variable Name</th>
<th>Basic Size</th>
<th>Design Space (±)</th>
</tr>
</thead>
<tbody>
<tr>
<td>b</td>
<td>10.4</td>
<td>2</td>
</tr>
<tr>
<td>e</td>
<td>49.3</td>
<td>5</td>
</tr>
<tr>
<td>$\theta_4$</td>
<td>42.646°</td>
<td>10°</td>
</tr>
</tbody>
</table>

The variations of point P in position and orientation are named $d\Delta x$, $d\Delta y$, and $d\Delta \phi$, respectively, in the following result tables. Fifteen experimental points are used to fit the surface to the nominal values as well as variations of the position and orientation of the point P. The VRS fitting has $R^2$ as 100% for all six surfaces. It gives the estimated equations for the variation surfaces in design space from the experimental values. Examining the surface equations for the nominal positions of point P, i.e., $\Delta x$ and $\Delta y$, shows very little effect from the second order terms, and the nominal orientation of point P,
i.e., $\Delta\phi$, is constant. The QV method from the VRS nominal fitting does not work in this case as the quadratic terms are negligible. Opt-Tol evaluates the tolerance sensitivity and the dependent variation at the corresponding nominal value at every analysis call in the algorithm. The maximum and minimum variations of the point P and their location in design space are listed in the following tables.

Table 4.10 Comparison of variation $d\Delta x$ from different methods

<table>
<thead>
<tr>
<th></th>
<th>VRS minimum</th>
<th>Real minimum</th>
<th>VRS maximum</th>
<th>Real maximum</th>
<th>Base</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d\Delta x$</td>
<td>0.5955271</td>
<td>0.5995524</td>
<td>0.7405299</td>
<td>0.740535</td>
<td>0.667891</td>
</tr>
<tr>
<td>$\Delta x$</td>
<td>-4.231708</td>
<td>-6.246806</td>
<td></td>
<td>-5.239254</td>
<td></td>
</tr>
<tr>
<td>b</td>
<td>8.4</td>
<td>8.4</td>
<td>12.4</td>
<td>12.4</td>
<td>10.4</td>
</tr>
<tr>
<td>e</td>
<td>44.3</td>
<td>44.3</td>
<td>54.3</td>
<td>54.3</td>
<td>49.3</td>
</tr>
<tr>
<td>$\theta_4$</td>
<td>42.646°</td>
<td>42.646°</td>
<td>42.646°</td>
<td>42.646°</td>
<td>42.646°</td>
</tr>
</tbody>
</table>

Table 4.11 Comparison of variation $d\Delta y$ from different methods

<table>
<thead>
<tr>
<th></th>
<th>VRS minimum</th>
<th>Real minimum</th>
<th>VRS maximum</th>
<th>Real maximum</th>
<th>Base</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d\Delta y$</td>
<td>0.2747579</td>
<td>0.274756</td>
<td>0.2784231</td>
<td>0.278421</td>
<td>0.276332</td>
</tr>
<tr>
<td>$\Delta y$</td>
<td>-56.556199</td>
<td>-60.011579</td>
<td></td>
<td>-58.283890</td>
<td></td>
</tr>
<tr>
<td>b</td>
<td>8.4</td>
<td>8.4</td>
<td>12.4</td>
<td>12.4</td>
<td>10.4</td>
</tr>
<tr>
<td>e</td>
<td>49.3</td>
<td>49.3</td>
<td>49.3</td>
<td>49.3</td>
<td>49.3</td>
</tr>
<tr>
<td>$\theta_4$</td>
<td>32.646°</td>
<td>32.646°</td>
<td>52.646°</td>
<td>52.646°</td>
<td>42.646°</td>
</tr>
</tbody>
</table>

Table 4.12 Comparison of variation $d\Delta \phi$ from different methods

<table>
<thead>
<tr>
<th></th>
<th>VRS minimum</th>
<th>Real minimum</th>
<th>VRS maximum</th>
<th>Real maximum</th>
<th>Base</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d\Delta \phi$</td>
<td>0.708778</td>
<td>0.708778</td>
<td>0.710593</td>
<td>0.710593</td>
<td>0.709387°</td>
</tr>
<tr>
<td>$\Delta \phi$</td>
<td>-0.000006</td>
<td>-0.000006</td>
<td>-0.000006</td>
<td>-0.000000°</td>
<td></td>
</tr>
<tr>
<td>b</td>
<td>10.4</td>
<td>10.4</td>
<td>10.4</td>
<td>10.4</td>
<td>10.4</td>
</tr>
<tr>
<td>e</td>
<td>49.3</td>
<td>49.3</td>
<td>49.3</td>
<td>49.3</td>
<td>49.3</td>
</tr>
<tr>
<td>$\theta_4$</td>
<td>32.646°</td>
<td>32.646°</td>
<td>52.646°</td>
<td>52.646°</td>
<td>42.646°</td>
</tr>
</tbody>
</table>

Figure 4.4 shows the design space cube and the predicted nominal design dimensions for the variation in the position and orientation of the point P. Opt-Tol as the
real surface method and VRS as the variation response surface predict the same location for nominal dimensions where the minimum and maximum objectives occurs.

Figure 4.4 Predicted dimensions for the remote positioning mechanism

The influence of each design variable on the maximum and minimum variations can be seen from the cube. The extreme variation is always at the boundary of the constrained design space. The angle $\theta_4$ has no effect on variation $d\Delta x$, length $e$ has no effect on $d\Delta y$, and length $b$ has no effect on $d\Delta \phi$. For minimum variations $d\Delta y$ and $d\Delta \phi$, angle $\theta_4$ should be a smaller value. For minimum $d\Delta x$, both $b$ and $e$ should be smaller values. The largest resulting variation of interest from design space can be identified to be the most conservative design [Robison 1989].

4.3 STACKED BLOCKS ASSEMBLY EXAMPLE

Problem Description

The stacked blocks assembly does not represent a real product, but it is used as an example of a multiple loop assembly and vector modeling techniques. The stacked blocks assembly consists of three parts: stepped base, block and cylinder. They are stacked together to form a closed assembly. Two resultant assembly dimensions have been selected for study: $u_1$ is the height of the cylinder and $u_3$ is the contacting location between cylinder and block. Both are resultant locations of point of contact. They are kinematic assembly dimensions and are affected by the nominal dimensions and tolerances of the three parts. Figure 4.5 shows the assembly and the vector loops for the stacked blocks assembly.
Figure 4.5 Stacked blocks assembly and vector loops

The vector model includes three closed loops. There are seven independent variables and nine dependent variables. All the manufactured variables together with nominal dimension and tolerances are listed in Table 4.13.

<table>
<thead>
<tr>
<th>Variable Name</th>
<th>Description</th>
<th>Basic Size</th>
<th>Tolerance (±)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a, ah</td>
<td>Cylinder radius</td>
<td>6.620 mm</td>
<td>.200 mm</td>
</tr>
<tr>
<td>f</td>
<td>Step width</td>
<td>3.905 mm</td>
<td>.125 mm</td>
</tr>
<tr>
<td>d</td>
<td>height of low step</td>
<td>4.060 mm</td>
<td>.150 mm</td>
</tr>
<tr>
<td>b</td>
<td>Block thickness</td>
<td>6.805 mm</td>
<td>.075 mm</td>
</tr>
<tr>
<td>c</td>
<td>Height of high step</td>
<td>10.675 mm</td>
<td>.125 mm</td>
</tr>
<tr>
<td>e</td>
<td>Step location</td>
<td>24.22 mm</td>
<td>.350 mm</td>
</tr>
</tbody>
</table>

The nominal kinematic assembly variables as well as calculated variations are listed in Table 4.14.
Table 4.14 Assembly dimensions for stacked blocks assembly

<table>
<thead>
<tr>
<th>Variable Name</th>
<th>Description</th>
<th>Basic Size</th>
<th>Variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>u1</td>
<td>Height of cylinder</td>
<td>18.71824 mm</td>
<td>0.259251</td>
</tr>
<tr>
<td>φ1</td>
<td>Dependent angle</td>
<td>74.72387°</td>
<td>0.47859°</td>
</tr>
<tr>
<td>u3</td>
<td>Location of contact point</td>
<td>8.670534 mm</td>
<td>0.227166</td>
</tr>
<tr>
<td>u2</td>
<td>Location of contact point</td>
<td>10.047707 mm</td>
<td>0.184374</td>
</tr>
<tr>
<td>φ4</td>
<td>Dependent angle</td>
<td>74.72387°</td>
<td>0.47859°</td>
</tr>
<tr>
<td>u4</td>
<td>Location of contact point</td>
<td>2.189437 mm</td>
<td>0.141061</td>
</tr>
<tr>
<td>φ3</td>
<td>Dependent angle</td>
<td>105.27613°</td>
<td>0.47859°</td>
</tr>
<tr>
<td>u5</td>
<td>Location of contact point</td>
<td>27.29654 mm</td>
<td>0.383594</td>
</tr>
<tr>
<td>φ2</td>
<td>Dependent angle</td>
<td>105.27613°</td>
<td>0.47859°</td>
</tr>
</tbody>
</table>

Results for Stacked Blocks Assembly from Design Space

The design space is shown in Table 4.15. It is spanned by dimensions b and c.

Table 4.15 Design space for stacked blocks assembly

<table>
<thead>
<tr>
<th>Variable Name</th>
<th>Basic Size</th>
<th>Design Space (±)</th>
</tr>
</thead>
<tbody>
<tr>
<td>b</td>
<td>6.805</td>
<td>1.5</td>
</tr>
<tr>
<td>c</td>
<td>10.675</td>
<td>2</td>
</tr>
</tbody>
</table>

Figure 4.6 Surface produced by VRS for du3
Figure 4.7 Surface produced by QV for du3

Figure 4.8 Surface produced by Opt-Tol for du3
The nominal surfaces, u1 and u3, and their variation surfaces, du1 and du3, have 100% R² in response surface fitting. The three surfaces produced by VRS, QV and Opt-Tol for the variation of u3, i.e., du3, are shown in Figure 4.6 to 4.8. They have the same trend.

The predicted location of design dimensions for estimated maximum and minimum du1 and du3 are the same for all three methods, except for the du1 from QV. They can be seen in the following.

![Diagram showing the predicted location of design dimensions for extreme du1 and du3](image)

Figure 4.9 Predicted location of design dimensions for extreme du1 and du3

The VRS and Opt-Tol methods consistently agree in estimating the design dimensions. It can be seen that variations for both du1 and du3 prefer the smaller value in c, the height of high step. In addition to the smaller step size, the thicker block produces less variation du1, while the basic block size produces less variation du3.

The main reason the VRS and QV methods give different result is that the QV variance only includes the tolerances of b and c, while the VRS variance includes all tolerance on all six manufacturing variation a through e listed in Table 4.13. Although the tolerances of all independent dimensions are known, the sensitivities of u1 and u3 to changes in a through e must be determined to compute variations du1 and du3. The sensitivities with respect to b and c are the only ones obtainable from the nominal surfaces.

If only the tolerances of b and c were considered in VRS, the prediction for dimensions would be the same as QV. Thus, the number of design dimensions allowed to vary has an effect on the dimension estimation. This is the principal limitation on the QV
method. Since the QV variance is determined from the nominal surface, the sensitivities for the other manufactured dimensions remain unknown.

The predictions from three methods for the two variations and their locations are listed in the following tables. The real variation values are listed in the row below the estimation for the comparison, which is calculated at the same location of design dimension as predicted by VRS or QV methods.

### Table 4.16 Comparison of variation du1 from different methods

<table>
<thead>
<tr>
<th></th>
<th>VRS minimum</th>
<th>QV minimum</th>
<th>Real minimum</th>
<th>VRS maximum</th>
<th>QV maximum</th>
<th>Real maximum</th>
<th>Base</th>
</tr>
</thead>
<tbody>
<tr>
<td>du1</td>
<td>0.2524895</td>
<td>0.08023695</td>
<td>0.25252433</td>
<td>0.2694392</td>
<td>0.08869559</td>
<td>0.26941267</td>
<td>0.259251</td>
</tr>
<tr>
<td>real</td>
<td>0.25252433</td>
<td>0.25374347</td>
<td>0.26941267</td>
<td>0.26840410</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>u1</td>
<td>19.770858</td>
<td>16.7168825</td>
<td>19.770858</td>
<td>17.6826389</td>
<td>20.8667695</td>
<td>17.6826389</td>
<td>18.71824</td>
</tr>
<tr>
<td>b</td>
<td>8.305</td>
<td>5.305</td>
<td>8.305</td>
<td>5.305</td>
<td>8.305</td>
<td>5.305</td>
<td>6.805</td>
</tr>
</tbody>
</table>

### Table 4.17 Comparison of variation du3 from different methods

<table>
<thead>
<tr>
<th></th>
<th>VRS minimum</th>
<th>QV minimum</th>
<th>Real minimum</th>
<th>VRS maximum</th>
<th>QV maximum</th>
<th>Real maximum</th>
<th>Base</th>
</tr>
</thead>
<tbody>
<tr>
<td>du3</td>
<td>0.2177057</td>
<td>0.04060012</td>
<td>0.21768599</td>
<td>0.2392741</td>
<td>0.0456565</td>
<td>0.239297</td>
<td>0.227166</td>
</tr>
<tr>
<td>real</td>
<td>0.21768599</td>
<td>0.21768599</td>
<td>0.23929663</td>
<td>0.23929663</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>u3</td>
<td>8.00051395</td>
<td>9.38103426</td>
<td>8.670534</td>
<td>8.670534</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>c</td>
<td>8.675</td>
<td>8.675</td>
<td>8.675</td>
<td>12.675</td>
<td>12.675</td>
<td>8.675</td>
<td>10.675</td>
</tr>
</tbody>
</table>

### 4.4 CLUTCH EXAMPLE

#### Problem Description

A one-way clutch transmits torque in a single direction. It is commonly used to start the engine of gas lawn mowers. Figure 4.10 shows the cross section of the assembly and the vector loop.

The hub is connected to the drive shaft of the engine and the spring loaded rollers can slide between the hub and ring. The clutch is locked when the ring rotates clockwise relative to the hub, so the torque can be applied to the shaft to start the engine. The hub
rotates faster than the ring after the engine is started, so the hub is unlocked and can rotate freely in a clockwise direction. The key to proper operation is the value of the pressure angle $\phi_1$, which is a resultant assembly variable. The optimum value for $\phi_1$ must be specified as an engineering requirement. The variation in $\phi_1$, i.e., $d\phi_1$ due to variations in a, c, and e must be minimized. Because of the geometric symmetry, the model only includes one fourth of the assembly. Also, from geometry, $\phi_1$ will always equal $\phi_2$.

![Figure 4.10 Clutch assembly and its vector loop](image)

<table>
<thead>
<tr>
<th>Variable Name</th>
<th>Description</th>
<th>Basic Size</th>
<th>Tolerance (±)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>Hub width</td>
<td>55.29 mm</td>
<td>.1 mm</td>
</tr>
<tr>
<td>c</td>
<td>Roller radius</td>
<td>11.43 mm</td>
<td>.01 mm</td>
</tr>
<tr>
<td>e</td>
<td>Ring radius</td>
<td>101.6 mm</td>
<td>.025 mm</td>
</tr>
</tbody>
</table>

Table 4.19 Assembly dimensions for Clutch

<table>
<thead>
<tr>
<th>Variable Name</th>
<th>Description</th>
<th>Basic Size</th>
<th>Variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>b</td>
<td>Roller contact</td>
<td>4.8105 mm</td>
<td>0.449451</td>
</tr>
<tr>
<td>$\phi_1$</td>
<td>Dependent angle</td>
<td>7.0184°</td>
<td>0.657877°</td>
</tr>
<tr>
<td>$\phi_2$</td>
<td>Dependent angle</td>
<td>7.0184°</td>
<td>0.657877°</td>
</tr>
</tbody>
</table>

Table 4.18 lists all the manufactured dimensions as well as the tolerances. The assembly variables together with nominal dimensions and computed variations are listed in Table 4.19.
Results for One-way Clutch from Design Space

Design Space

The estimations are based on the design space spanned by the hub width a and roller radius c. The ring radius will be held constant. In this case, only considering dimensional constraints in a and c can violate the feasibility constraint and make the dependent variable $\phi_1$ have no solution. Thus, in addition to size limits on a and c, physical limits on the pressure angle $\phi_1$ must be set. This results in a design space which is not square.

The constant dependent angle $\phi_1$ appears as a straight line with negative slope in the design space as

$$c = -\frac{a}{2(\cos \phi + 1)} + \frac{e \cos \phi}{2(\cos \phi + 1)}$$

(4.1)

This relationship is used for the physical constraint. The design space is constrained by dimensional and physical constraints. The dependent angle $\phi_1$ is limited to the range of $3.5^\circ$ to $14^\circ$; it is near the current nominal $7^\circ$ and well within the feasibility limits. The line representing $7^\circ$ is very close to the boundary constraint line of $3.5^\circ$.

![Figure 4.11 Constrained design space for clutch case](image)

Figure 4.11 Constrained design space for clutch case
The choice of the fitting points is not based on an explicit criterion: the coverage should start on the boundary of the factor space. This is from the following guide line: the best way to learn about the slope is to go to the extremes of the factor space [Kennard et al. 1969]. The fitting points are the corner, boundary and nominal points. At least three levels are needed for each independent variable to make enough data points. The data obtained are used for the quadratic fitting as the estimation for the variation of the dependent angles. The constraint design space is shown in Figure 4.11.

Contour Plots

For the estimation of variation of the dependent angle, the regression result shows that s, i.e., estimated standard derivation about regression, is 0.2323 and $R^2$ is 96.3%. Approximately 96.3% of the variability in $\phi_1$ is explained by the explanatory variables a and c through the quadratic equations.

The contours for $d\phi_1$, the variation of the dependent angle, over a section of design space are plotted in Figure 4.12.

![Approximate Angle Variation I](image)

Figure 4.12 Contours of constant variation $d\phi_1$ from VRS

The horizontal coordinate is the hub thickness a and the vertical coordinate is the roller radius c. It shows that the variation has a smaller value when the nominal angle $\phi_1$ approaches the larger limit 14°, and has larger variation when the nominal angle is close to
the smaller limit 3.5°. The slope of the surface also changes greatly when the nominal angle nears the smaller angle. Lines of constant variation form nearly parallel lines, which are approximately parallel to the boundary constraints.

Since an explicit expression for the angle $\phi_1$ can be obtained in this example, the dependent angle constraints can be calculated. Together with the independent constraints, they are defined as design functions. The real variation $d\phi_1$ can be calculated and compared with the regression result. The real angle variation is shown in Figure 4.13.

**Real Angle Variation I**

![Contour plot](image)

**Figure 4.13** Contours of real constant variation $d\phi_1$

It looks as if the range near the current nominal 7°, very close to 3.5° boundary, is in the area of steepest descent. This means the area is very sensitive to the values a and c. The more close to 14° the angle is, the less sensitive it will be. The slope changes more slowly in the range near 14° and the area is more "flat." Design in this area will be more robust to variation, but may not perform as well.

The contours of the estimated $d\phi_1$, shown in Figure 4.12, follow a similar trend, but the slope change is more moderate compared with the contours of real $d\phi_1$, shown in Figure 4.13. For certain variations of the dependent angle, there are different designs that can be chosen. For less variation only, there are certain ranges of the nominal value that can be chosen. The minimum variation of the angle occurs at the maximum nominal angle.
The accuracy can be observed by plotting contours of the ratio of real over approximate value of $d\phi_1$. It is shown in Figure 4.14.

Real/appro. Angle Variation I

![Contour Plot]

Figure 4.14 Contours of the ratio of real over approximate

It can be seen from the plot that the prediction is rather accurate at the boundary and less accurate near the center. The estimation is reasonable. It helps the designer to see the variation behavior in the design space and to consider tolerance in the early design stage, before final dimensions have been set.

In Figure 4.13, the function value is calculated at numerous points and the optimization algorithm can find the constrained extremes. It requires explicit constraints for the dependent variable. However, if the model has more variables and more loops, the explicit form for dependent variables would be very difficult to obtain.

Statistical variation surface fitting gives an estimate of $d\phi_1$ over the design space. Each point used in the fit requires a complete analysis. The values are used in the regression algorithm. It requires much less computation than real variation analysis.

**Extreme Points**

The maximum and the minimum points for the real and approximate surfaces can both be obtained by optimization techniques.
<table>
<thead>
<tr>
<th></th>
<th>VRS minimum</th>
<th>Real minimum</th>
<th>VRS maximum</th>
<th>Real maximum</th>
<th>Base</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d\phi_1$</td>
<td>0.297741$^\circ$</td>
<td>0.331301$^\circ$</td>
<td>1.533659$^\circ$</td>
<td>1.429529$^\circ$</td>
<td>0.657877$^\circ$</td>
</tr>
<tr>
<td>real</td>
<td>0.331301$^\circ$</td>
<td></td>
<td>1.391928$^\circ$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\phi_1$</td>
<td>14$^\circ$</td>
<td>14$^\circ$</td>
<td>3.5$^\circ$</td>
<td>3.5$^\circ$</td>
<td>7.0184$^\circ$</td>
</tr>
<tr>
<td>a</td>
<td>53.539</td>
<td>53.539</td>
<td>47.433</td>
<td>43.414</td>
<td>55.29</td>
</tr>
<tr>
<td>c</td>
<td>11.43</td>
<td>11.43</td>
<td>13.507</td>
<td>14.513</td>
<td>11.43</td>
</tr>
<tr>
<td>real/prediction</td>
<td>1.113</td>
<td></td>
<td></td>
<td></td>
<td>0.908</td>
</tr>
</tbody>
</table>

Comparing locations of a and c for extremes of $d\phi_1$, the prediction for the minimum $d\phi_1$ is effective, since the design dimensions a and c for the both estimated and the real methods are consistent. The design dimensions a and c for maximum variation for two methods are not the same, but both occur along 3.5$^\circ$ boundary. The real $d\phi_1$ values at the design dimensions corresponding to the predictions are put in the third row. It is used to compare. The ratio of real over estimate is in the range of 0.9 to 1.1.

The minimum value for the variation $d\phi_1$ occurs at the 14$^\circ$ boundary with the maximum value of a along that boundary. The maximum variation $d\phi_1$ from real value occurs at the highest left hand corner on the 3.5$^\circ$ boundary.

In the contours, the nominal angle 7$^\circ$ used in the basic design occurs close to the 3.5$^\circ$ boundary. This is close to the steepest decent area in the contour where variation increases. The designer needs to select proper nominal dimensions based on performance and variation.

The special difficulty for the clutch is to satisfy kinematic constraints. It limits the Opt-Tol method. If the search falls in the range where the dependent variables have no solution, the iterative process cannot continue.

Other points can be chosen to improve fitting result, such as middle point, weighted middle point, point with maximum error in the estimation evaluation, etc. For constrained spaces, the choice of points is still an open issue.
4.5 SUSPENSION EXAMPLES

Suspension system is a mechanism in automobiles, which attaches the wheel to the body and allows the wheel to move relative to the body. The system kinematics is greatly influenced by the geometry of the mechanism, i.e., the location and orientation of the joints and the lengths of the linkage elements. It affects the system performance. Double Arm suspension and McPherson strut suspension are widely used independent suspension systems.

When a tire is going over a bump, sideways motion of the tire occurs. This sideways motion is called scrub. This is an important parameter affecting performance. Scrub will cause the tire to drag and wear. The designer prefers it to be as small as possible.

Double Arm Suspension

![Diagram of Double Arm Suspension System]

Figure 4.15 Model of the Double Arm suspension system and basic vector loops

A Double Arm suspension is also called short-long arm suspension and uses two control arms of unequal lengths. The inner ends of the control arms are attached to the body and allowed to pivot. The coupler link connects the two control arms. The wheel
axle is rigidly attached to the coupler link. Spherical joints allow the coupler to rotate about a vertical axis. This allows steering to take place as well as vertical suspension motion. This means that for a given position of both control arms, the rotation gives an infinite solution of the orientation of the coupler about the line segment rkp without moving the control arms.

A subset of the front suspension system is used for analysis. Only vertical motion in a plane is considered. Steering rotations are ignored, reducing the assembly to a 2-D mechanism with one degree of freedom. The vector model and the basic loops are shown in Figure 4.15.

To model the scrub, two different positions of the suspension are considered, the position at rest, i.e., equilibrium position, and the position at maximum tire displacement. The road shock for the suspension assembly is 2.5 in, which is the maximum tire displacement. The largest scrub occurs then. The vector loops in Figure 4.16 show both positions simultaneously. The scrub appears as a variable in the open loops, which is u in the plot. This model considers the kinematics due to outside disturbances, as well as manufacturing and assembly variations. All the dependent joint angles must be obtained by solution of the corresponding vector loop equations.

Figure 4.16 Vector loops for analyzing the scrub for Double Arm suspension
This vector model consists of four closed loops and one open loop. It produces thirteen nonlinear equations that can be used to solve all dependent variables. The numerical method is used to find all the nominal values of the assembly variables. Once the nominal geometry of the system is obtained, the linearized method is used to solve for the variations. Table 4.21 lists the manufactured variables [Beard 1992] and the tolerances. Table 4.22 lists all the dependent variables and the variations calculated by the linearized method.

Table 4.21 Manufactured dimensions for Double Arm suspension

<table>
<thead>
<tr>
<th>Variable Name</th>
<th>Description</th>
<th>Basic Size</th>
<th>Tolerance (±)</th>
</tr>
</thead>
<tbody>
<tr>
<td>rc</td>
<td>Length of EC</td>
<td>3</td>
<td>1 mm</td>
</tr>
<tr>
<td>θ1</td>
<td>Joint angle</td>
<td>84.46°</td>
<td>1°</td>
</tr>
<tr>
<td>rla</td>
<td>Length of lower arm</td>
<td>15</td>
<td>1 mm</td>
</tr>
<tr>
<td>rf2</td>
<td>Distance between joints</td>
<td>15.53</td>
<td>1 mm</td>
</tr>
<tr>
<td>θ4</td>
<td>Joint angle</td>
<td>94°</td>
<td>1°</td>
</tr>
<tr>
<td>rua</td>
<td>Length of upper arm</td>
<td>8</td>
<td>1 mm</td>
</tr>
<tr>
<td>rkp</td>
<td>Length of DC</td>
<td>18.5</td>
<td>1 mm</td>
</tr>
<tr>
<td>θ5</td>
<td>Joint angle</td>
<td>108.7°</td>
<td>1°</td>
</tr>
</tbody>
</table>

Table 4.22 Assembly dimensions for Double Arm suspension

<table>
<thead>
<tr>
<th>Variable Name</th>
<th>Description</th>
<th>Basic Size</th>
<th>Variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>ry1</td>
<td>Tread width at rest</td>
<td>28.48129</td>
<td>0.29905</td>
</tr>
<tr>
<td>φ1</td>
<td>Variable angle</td>
<td>90.36334°</td>
<td>1.46741°</td>
</tr>
<tr>
<td>φ2</td>
<td>Variable angle</td>
<td>11.31908°</td>
<td>1.3097°</td>
</tr>
<tr>
<td>φ3</td>
<td>Variable angle</td>
<td>17.22243°</td>
<td>0.23842°</td>
</tr>
<tr>
<td>φ4</td>
<td>Variable angle</td>
<td>140.1524°</td>
<td>2.29689°</td>
</tr>
<tr>
<td>φ5</td>
<td>Variable angle</td>
<td>68.45097°</td>
<td>1.82453°</td>
</tr>
<tr>
<td>ry2</td>
<td>Displaced tread width</td>
<td>28.46897</td>
<td>0.2837</td>
</tr>
<tr>
<td>φ6</td>
<td>Variable angle</td>
<td>95.09359°</td>
<td>1.3299°</td>
</tr>
<tr>
<td>φ7</td>
<td>Variable angle</td>
<td>15.69586°</td>
<td>1.19511°</td>
</tr>
<tr>
<td>φ8</td>
<td>Variable angle</td>
<td>26.32945°</td>
<td>0.29125°</td>
</tr>
<tr>
<td>φ9</td>
<td>Variable angle</td>
<td>159.6575°</td>
<td>2.63943°</td>
</tr>
<tr>
<td>φ10</td>
<td>Variable angle</td>
<td>53.67613°</td>
<td>1.92474°</td>
</tr>
<tr>
<td>u</td>
<td>Tire scrub</td>
<td>0.012313</td>
<td>0.043671</td>
</tr>
</tbody>
</table>
McPherson Strut Suspension

McPherson strut suspension uses an oversized, telescoping shock absorber, called the strut, as part of the structure. The upper end of the strut is connected to the body by a spherical joint and a cylindrical joint connects the coupler link to the upper portion of suspension. The lower end of the coupler link is attached to the car by the lower control arm. The axle is rigidly attached to the coupler link, or lower potion of the McPherson strut suspension. Figure 4.17 shows the model and basic vector loops for McPherson strut suspension system.

Figure 4.17 Model of the McPherson strut suspension system and basic vector loops

For scrub analysis, a vector model with two positions is needed. Figure 4.18 shows the model vector loops for the McPherson strut suspension system. This vector model also produces four closed loops and one open loop and thirteen nonlinear equations.
Figure 4.18 Vector loops for analyzing the scrub for McPherson strut suspension

The manufactured variables are listed in Table 4.23 [Beard 1992] together with the tolerances. The nominal values and variation for dependent variables are obtained by the same methods as discussed for the Double Arm suspension system. They are listed in Table 4.24 for the nominal design.

Table 4.23 Manufactured dimensions for McPherson strut suspension

<table>
<thead>
<tr>
<th>Variable Name</th>
<th>Description</th>
<th>Basic Size</th>
<th>Tolerance (±)</th>
</tr>
</thead>
<tbody>
<tr>
<td>rc</td>
<td>Length of EC</td>
<td>9.4</td>
<td>1 mm</td>
</tr>
<tr>
<td>θ1</td>
<td>Joint angle</td>
<td>62.36°</td>
<td>1°</td>
</tr>
<tr>
<td>ra</td>
<td>Length of arm</td>
<td>12.5</td>
<td>1 mm</td>
</tr>
<tr>
<td>rf2</td>
<td>Distance between joints</td>
<td>28</td>
<td>1 mm</td>
</tr>
<tr>
<td>θ4</td>
<td>Joint angle</td>
<td>120°</td>
<td>1°</td>
</tr>
<tr>
<td>θ6</td>
<td>Joint angle</td>
<td>21.3°</td>
<td>1°</td>
</tr>
<tr>
<td>rp</td>
<td>Length of DC</td>
<td>4.18</td>
<td>1 mm</td>
</tr>
<tr>
<td>θ5</td>
<td>Joint angle</td>
<td>90°</td>
<td>1°</td>
</tr>
</tbody>
</table>
Table 4.24 Assembly dimensions for McPherson strut suspension

<table>
<thead>
<tr>
<th>Variable Name</th>
<th>Description</th>
<th>Basic Size</th>
<th>Variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>ry1</td>
<td>Tread width at rest</td>
<td>27.93647</td>
<td>0.25914</td>
</tr>
<tr>
<td>φ1</td>
<td>Variable angle</td>
<td>89.99704°</td>
<td>1.91725°</td>
</tr>
<tr>
<td>φ2</td>
<td>Variable angle</td>
<td>28.14293°</td>
<td>2.72652°</td>
</tr>
<tr>
<td>φ3</td>
<td>Variable angle</td>
<td>0.505888°</td>
<td>1.0927°</td>
</tr>
<tr>
<td>rs1</td>
<td>Variable length</td>
<td>24.04462</td>
<td>0.38885</td>
</tr>
<tr>
<td>φ4</td>
<td>Variable angle</td>
<td>36.33704°</td>
<td>0.18053°</td>
</tr>
<tr>
<td>ry2</td>
<td>Displaced tread width</td>
<td>27.72398</td>
<td>0.31388</td>
</tr>
<tr>
<td>φ5</td>
<td>Variable angle</td>
<td>90.12694°</td>
<td>2.07852°</td>
</tr>
<tr>
<td>φ6</td>
<td>Variable angle</td>
<td>16.83404°</td>
<td>3.0642°</td>
</tr>
<tr>
<td>φ7</td>
<td>Variable angle</td>
<td>10.9329°</td>
<td>1.24421°</td>
</tr>
<tr>
<td>rs2</td>
<td>Variable length</td>
<td>21.54426</td>
<td>0.40944</td>
</tr>
<tr>
<td>φ8</td>
<td>Variable angle</td>
<td>36.46694°</td>
<td>0.1918°</td>
</tr>
<tr>
<td>u</td>
<td>Tire scrub</td>
<td>0.212491</td>
<td>0.073675</td>
</tr>
</tbody>
</table>

Design Space

For both suspension systems, the constraints for the nominal independent dimensions are only size constraints and are far from unfeasible space for the dependent variables. Three design variables are chosen to be changed from the nominal design. The first design variable is the length of the lower control arm rla. The second is the length rf2 of the frame between the two joints. The first joint connects the lower control arm to frame. For Double Arm suspension, the second joint connects the upper control arm to the frame. For McPherson strut suspension, the second joint connects the McPherson strut to the frame. The third design variable is the angle φ4 controlling orientation of rf2. The design spaces for both are shown in Table 4.25 and 4.26. The dependent variables are always solvable in this region.

Table 4.25 Design space for Double Arm suspension

<table>
<thead>
<tr>
<th>Variable Name</th>
<th>Description</th>
<th>Basic Size</th>
<th>Design Space (±)</th>
</tr>
</thead>
<tbody>
<tr>
<td>rla</td>
<td>Length of lower arm</td>
<td>15</td>
<td>2</td>
</tr>
<tr>
<td>rf2</td>
<td>Distance between joints</td>
<td>15.53</td>
<td>2</td>
</tr>
<tr>
<td>rf2ang</td>
<td>Orientation of rf2</td>
<td>266°</td>
<td>10°</td>
</tr>
</tbody>
</table>
Table 4.26 Design space for McPherson strut suspension

<table>
<thead>
<tr>
<th>Variable Name</th>
<th>Description</th>
<th>Basic Size</th>
<th>Design Space (±)</th>
</tr>
</thead>
<tbody>
<tr>
<td>ra</td>
<td>Length of arm</td>
<td>12.5</td>
<td>2</td>
</tr>
<tr>
<td>rf2</td>
<td>Distance between joints</td>
<td>28</td>
<td>2</td>
</tr>
<tr>
<td>rf2ang</td>
<td>Orientation of rf2</td>
<td>240°</td>
<td>10°</td>
</tr>
</tbody>
</table>

A central composite design with α=1 is used to study the effects of the three design factors. Only three levels for each factor are needed to prepare data for the surface fitting. Here, the “star points” in the central composite design correspond to the center of the six sides of the cubes. This design is called a face-centered-cube design. Fifteen sets of design variable values are used for the experiment. The nonlinear solver and linearized method are used to determine nominal value of the dependent variables and corresponding variation resulting at each experimental point. The second order response surface can be fitted from the fifteen set of data for nominal scrub and the variation of the scrub.

Results and Comparison

The scrub is approximated near a point of interest, that is the nominal design. It has been analyzed through three different methods, i.e., VRS, QV and Opt-Tol. Here, eight independent tolerances are considered. The scrub is the variable in the open loop. It is the function of the dependent variables of the closed loops also. Its variation is derived following the steps in Chapter 2.

The $R^2$ values in VRS for both designs are shown in the following table. Here, $u$ is the scrub and $du$ is variation of the scrub. The response surface can explain more than 95% of the variability and estimated standard derivation less than 0.08.

Table 4.27 Surface fitting parameters for Double Arm and McPherson strut suspensions

<table>
<thead>
<tr>
<th></th>
<th>u (DA)</th>
<th>du (DA)</th>
<th>u (Mc)</th>
<th>du (Mc)</th>
</tr>
</thead>
<tbody>
<tr>
<td>s</td>
<td>0.07464</td>
<td>0.005774</td>
<td>0.05886</td>
<td>0.01204</td>
</tr>
<tr>
<td>$R^2$</td>
<td>99.4%</td>
<td>99.4%</td>
<td>99.7%</td>
<td>98.3%</td>
</tr>
<tr>
<td>$R^2$(adj)</td>
<td>98.3%</td>
<td>98.3%</td>
<td>99.2%</td>
<td>95.4%</td>
</tr>
</tbody>
</table>

The response surfaces for the scrub and variation of the scrub for Double Arm suspension are shown in Figures 4.19 and 4.20. They show how the scrub behaves in
different suspension systems when the nominal values of the independent variable change anywhere in the design space.

Figure 4.19 Response surface for u (Double Arm suspension)

Figure 4.20 Response surface for du (Double Arm suspension)
Corresponding response surfaces for the McPherson strut suspension are shown as follows.

Figure 4.21 Response surface for u (McPherson strut suspension)

Figure 4.22 Response surface for du (McPherson strut)
The stationary points for the quadratic nominal of scrub are outside the design space for both Double Arm and McPherson strut suspension systems. Use of these points as designs will be excluded by practical considerations.

Based on three surface methods, the scrub and the scrub with variation are set as objectives and their maximum and minimum are obtained. The complete results are listed in Appendix Table A.1 to A.4. The estimated assembly results can be represented in two different ways, i.e., by the location of the design dimensions on design cubes and by bar charts of the values.

Design cube

For three design variables, the permissible range may be represented as a cube in a 3-D design space. By design dimension estimation, the dimensions corresponding to maximum and minimum of the objective, such as variation of the scrub or sum of the scrub and its variation, are obtained. The locations in design space are found by each of the three methods and plotted in the design cube. The difference of the parameters provides criteria for comparing the prediction accuracy for the design variables.

For the variation of the Double Arm suspension, the maximum occurs on an edge of the cube, as shown in Figure 4.23. In the design space cube, all three predictions are found on the same edge. The maximum point from VRS is closer to the Real than QV point.

Also of interest is the largest and smallest value of the scrub occurring in design space. By adding the nominal scrub u and its variation du, the expected range of scrub can be estimated over design space.

For the sum of scrub and its variation, i.e., u+du, the VRS and QV predict the same design dimensions for the maximum and minimum points. The maximum points are very close to the real maximum point, which is the greatest concern for the designer. The minimum points have some distance between them, as shown in the plot. As the scrub is the main contributor in this objective function and predicted directly by quadratic fitting, the sum of the scrub and its variation has better prediction than only the variation. This can be seen from the overlap of the VRS and QV points in Figure 4.24.
Figure 4.23 Location of minimum and maximum du points (Double Arm)

Figure 4.24 Location of minimum and maximum u+du points (Double Arm)
Figure 4.25 Location of minimum and maximum du points (McPherson strut)

Figure 4.26 Location of minimum and maximum u+du points (McPherson strut)
It can be seen from Figure 4.25 that the estimated extremes of variation for McPherson strut suspension are not closely grouped. VRS estimates still predict design dimension as being very close to real variation from Opt-Tol method, while the QV method shows large error in prediction.

From the Figure 4.26, VRS gives the same prediction for the design dimension as QV if the objective is scrub plus variation, since the scrub is dominant.

Objective values

![Graph showing objective values for VRS, QV, and REAL.](image)

**Figure 4.27** du for Double Arm and McPherson strut suspension designs

When comparing objective values, we examine the extreme value of assembly parameters, rather than the location of the extremes in the design space. The estimated objectives, i.e., variation of scrub and scrub with variation at predicted dimensions, are plotted as bar charts. The results for the Double Arm and McPherson strut suspension systems are plotted together. The size of design space is the same for all three methods. From the bar chart, the dotted bars behind the VRS and the QV are the real variation values at the corresponding design dimensions. The relative error can be seen by comparison with the background bars. The real maximum and minimum objectives are plotted as a third set.
of bars, which is used as a standard reference. The comparison of alternative designs, i.e., Double Arm and McPherson strut suspensions, and three surface methods, i.e., VRS, QV, and Opt-Tol, can be seen from the same plot.

The bar chart for scrub variation in Figure 4.27 shows that VRS du values are not far from the real du value.

![Bar chart](image)

**Figure 4.28** u+du for Double Arm and McPherson suspensions

Figure 4.28 compares the values found for the sum of the scrub and the its variation. Since the magnitude of the scrub u is dominant compared with the variation of the scrub du, the bar chart for the scrub with variation shows QV is also an acceptable estimate.

The above study only considers the requirement for the performance of the suspension systems with respect to the scrub. The design dimensions have assigned nominal and tolerance values. The cube and bar charts include information about two alternative designs as viewed by three methods. Comparing the alternative designs, the Double Arm suspension has less scrub and less scrub plus variation in the range of the design space than the McPherson strut suspension. Double Arm suspension appears to be
a better design. Comparing estimation methods, VRS gives better prediction than the QV. Opt-Tol is the most accurate method for the analysis.

4.6. SUMMARY OF SURFACE METHODS

Three methods have been applied to six different case studies. The predicted design dimension for the extreme variations and the predicted variations are obtained in each case.

For the objective value, the relative accuracy is defined as

\[
\text{Relative accuracy} = 1 - \frac{|\text{True value} - \text{Estimated value}|}{\text{True value}}
\]

(4.2)

[Figliola et al. 1991]. The following tables can be built from the results of the case studies. The results from Opt-Tol are used as a standard to compare. VRS has more than 86% average relative accuracy for the predicted variations. The limited number of tolerances which can be included in QV method decreases its accuracy.

<table>
<thead>
<tr>
<th>Table 4.28 Relative accuracy for predicted variations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of parameter</td>
</tr>
<tr>
<td>----------------------</td>
</tr>
<tr>
<td>Number of parameter</td>
</tr>
<tr>
<td>Relative accuracy</td>
</tr>
</tbody>
</table>

For the predicted design dimensions, the estimate error is calculated by

\[
\text{Estimate error} = \frac{|\text{Dimension at real extrem} - \text{Dimension at estimated extrem}|}{\text{Range of design space}}
\]

(4.3)

Table 4.29 shows that VRS has average error less than 5% and QV has average error less than 19% for the predicted dimensions at the extremes.

<table>
<thead>
<tr>
<th>Table 4.29 Estimate error of predicted dimensions at extremes</th>
</tr>
</thead>
<tbody>
<tr>
<td>At the point of</td>
</tr>
<tr>
<td>Number of parameter</td>
</tr>
<tr>
<td>----------------------</td>
</tr>
<tr>
<td>Number of parameter</td>
</tr>
<tr>
<td>Estimate error</td>
</tr>
</tbody>
</table>
Table 4.30 gives a general summary and comparison. In the table, Nspace represents the dimension of the design space and Ntol represents the number of tolerances included.

<table>
<thead>
<tr>
<th>Table 4.30 Summary and comparison of three methods for variation analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nominal surface</td>
</tr>
<tr>
<td>-----------------</td>
</tr>
<tr>
<td>not required, but may be useful for design evaluation</td>
</tr>
<tr>
<td>Variation surface</td>
</tr>
<tr>
<td>Maximum number of points needed</td>
</tr>
<tr>
<td>Nonlinear solution for nominal</td>
</tr>
<tr>
<td>Linearized solution for variation</td>
</tr>
<tr>
<td>Ntol &amp; Nspace</td>
</tr>
<tr>
<td>Find extreme values by</td>
</tr>
<tr>
<td>Accuracy</td>
</tr>
</tbody>
</table>

The summary describes the procedures for the different methods. In VRS, at every experimental point, the nominal values of the dependent variables are solved for the assembly through the nonlinear system of equations, and the linearized method is used to calculate the sensitivities and the variations where all independent tolerances are included. Then, the regression model is fitted to the calculated variations of the dependent variable of interest. The resulting surface provides estimates of the variation parameters.
In contrast, by the QV method, the nominal fitting is required as the first step, where the regression equation for the nominal of the dependent variable is expressed in terms of the independent variables of interest. If adequate, the derivative of the nominal surface is taken and the variance function can be obtained. The error in predicting variation by the QV method is due to the limitation that the dimension of the design space, i.e., the number of independent factors, equals the number of the independent tolerances.

It appears that the VRS gives a much more accurate estimation than QV. This can be seen in the case studies from both the location of the design dimensions and estimated response values. Here, the design dimensions and variations from Opt-Tol are used as the standard values. It is the best method for accurate design surfaces. It requires the general optimization package and interface to connect the optimization algorithm to the tolerance analysis. Opt-Tol can handle modeling, analyzing and optimizing for the tolerance analysis. It is an effective tool. The VRS and QV methods, although they are less accurate, they require much less computation. They are suitable for approximating design performance in the conceptual stages of design.
Chapter 5

A NEW EFFECTIVE APPROACH - VARIATION POLYGON METHOD

Tolerance sensitivity is very important in assembly tolerance analysis. It represents the degree of sensitivity of an assembly variable relative to a manufacturing variable and it is used to predict the accumulated variations. The tolerance sensitivity can be calculated by taking derivatives of the constraint equations for geometric sensitivity and applying matrix algebra. The calculation procedure, being numerical, prevents a complete understanding of the nature of tolerance sensitivity. As design alternatives are considered, any change from current nominal dimensions requires the complete analysis to be repeated. Such detailed data analysis is not so convenient in the early design stage, as nominal changes at the design stage are most likely to occur. The physical information, such as direction and magnitude of variation in the assembly, and the relationship between variations of dependent and the independent variables cannot be clearly seen through a data analysis approach.

The geometric nature of tolerance sensitivity, variation mechanism and kinematic adjustment have not been emphasized before. No work has been done to express tolerance sensitivity of mechanical assemblies in closed form, geometric relationships. The nature of tolerance sensitivity needs to be better understood. If tolerance sensitivity can be represented as a relationship instead of as numeric data, it will greatly facilitate tolerance design at an early stage. It becomes a very convenient way to improve the nominal design as well as the tolerance design. This approach will help the designer to better understand variation relationships and tolerance sensitivity. The more understandable the mechanism of tolerance sensitivity becomes, the more effective quality control will be.

This chapter presents a new concept for analyzing assembly tolerance sensitivity and variations. The approach is both geometric and analytical. The geometric nature of tolerance sensitivity and the relationships between the independent and dependent variables are emphasized in the analysis.

For a better understanding of the nature of the tolerance sensitivity, an attribute in three different domains is presented in the first section. Some important qualities of sensitivity will be discussed for the geometric nature. The need for a better way to analyze tolerance sensitivity becomes obvious. A new effective tolerance analysis method, the variation polygon, is presented through an example.
5.1 COMMON ATTRIBUTE

Tolerance sensitivity matrix represents the relationship between dependent and independent variations. Analogies to tolerance sensitivity have been found in generalized motion and force transformation. They are presented in this section. Three different domains: variation, motion and force, will be analyzed. An attribute appears in each domain as an important characteristic. They have different functions.

**Variation**

In assembly tolerance analysis, [S] represents the matrix of tolerance sensitivities as discussed in Chapter 2. It is derived from the first order Taylor's series expansion of the constrained equations. The tolerance sensitivity can be expressed in matrix form as discussed in Chapter 2.

\[
[S] = -[B]^{-1}[A]
\]  \hspace{1cm} (5.1)

or

\[
[S] = \left[ \frac{dU}{dX} \right]
\]  \hspace{1cm} (5.2)

Tolerance sensitivity describes the relationship of variations between dependent variables and independent variables. It shows how the manufacturing variations \(dX\) are transformed into an assembly as errors in the adjustable dimensions \(dU\). The magnitude of each component in \([S]\) tells how sensitive the dependent variation is to the independent variation in the assembly. The tolerance sensitivity matrix is evaluated at the nominal dimensions of the assembly and used in error analysis. The product of the corresponding tolerance sensitivity, multiplied by the variation of the independent variable, tells how much variation of the dependent variable is contributed by this independent variation. Examining the tolerance sensitivity and percent contribution allows the tolerance to be reasonably assigned for the independent variables, in keeping with the cost and the manufacturing consideration.

A linear assembly is a one-dimensional tolerance stacking case. The tolerance sensitivity for each dimension is either +1 or -1, depending on the directions of dependent and independent variables of interest. A zero element means that there is no relationship. \([S]\) does not appear in the tolerance accumulation expressions. Worst case and statistical analysis for a linear assembly will have the expression:
\[ \{dU\} = \sum |dx_i| \]  \hspace{2cm} (5.3)

\[ \{dU\} = \sqrt{\sum [dx_i]^2} \]  \hspace{2cm} (5.4)

Nonlinear assemblies occur in 2-D and 3-D tolerance stacking cases. Weighted summations of the independent variations are required to account for variations. In worst case or statistical analysis, only magnitudes of the tolerance sensitivity are used.

\[ \{dU\} = \sum \left| \frac{\partial u_i}{\partial x_i} \right| dx_i \]  \hspace{2cm} (5.5)

\[ \{dU\} = \sqrt{\sum \left( \frac{\partial u_i}{\partial x_i} \right)^2 dx_i} \]  \hspace{2cm} (5.6)

An assembly system should be robust to variation. In other words, the system is insensitive to the variation in a “noisy” manufacturing environment. Once the tolerance sensitivities are obtained, the more sensitive dimension may be identified. They have the most critical influence on the design quality, and their tolerances can be tightened. The less sensitive relationships allow the designer to relax tolerances for lower manufacturing cost. The designer is more confident in assigning the tolerances, if the information about tolerance sensitivity is available.

The Equation 5.2 shows that tolerance sensitivity is a variation ratio. It can be used to find the dependent variation, if the independent variation is known. It is variation independent and it is decided by nominal design dimensions. Whenever the dimensions of the design change, the tolerance sensitivity will change. If the relationship between the nominal dimensions and sensitivity can be determined, it can help the nominal design, as well as tolerance design.

**Motion**

For a single degree closed mechanism, \( x \) is a known driving variable described by a Lagrangian coordinate as a primary (or generalized) coordinate. The variables \( u_j \) (\( j = 1, 2, ..., N \)) are unknown described by \( N \) secondary coordinates. They are related by constraints of loop closure.

\[ L_i (u_1, ..., u_N, x) = 0 \hspace{1cm} i = 1, 2, 3, ..., N \]  \hspace{2cm} (5.7)
Taking the derivative of equations, with respect to time, yields the velocity equations as $N$ equations of linear constraint of the form

$$\sum_{j=1}^{N} \frac{\partial L_i}{\partial u_j} \dot{u}_j = -\frac{\partial L_i}{\partial x} \dot{x} \tag{5.8}$$

Introducing

$$k_j = \frac{\dot{u}_j}{\dot{x}} \tag{5.9}$$

Equation 5.8 becomes

$$\sum_{j=1}^{N} \frac{\partial L_i}{\partial u_j} k_j = -\frac{\partial L_i}{\partial x} \tag{5.10}$$

In matrix notation

$$[B] [K] = -[A] \tag{5.11}$$

Where

$$[B] = \begin{bmatrix} \frac{\partial L_i}{\partial u_j} \end{bmatrix}, \text{ an } N \times N \text{ matrix,}$$

$$[A] = \begin{bmatrix} \frac{\partial L_i}{\partial x} \end{bmatrix}, \text{ an } N \times 1 \text{ matrix,}$$

Both $[A]$ and $[B]$ matrices are functions of the instantaneous configuration of the system.

$$[K] = -[B]^{-1}[A], \text{ an } N \times 1 \text{ matrix} \tag{5.12}$$

The elements in $[K]$ as shown in Equation 5.9 are defined as the velocity ratio or influence coefficient [Paul 1979]. $[K]$ is a column vector for single degree freedom system with $N$ element. From the equation, all $k_j$ are independent of the specific driving velocity of the moving part. It can be used to find the unknown secondary velocity $\dot{u}_j$, if the driving velocity $\dot{x}$ is known

$$\dot{u}_j = k_j \dot{x} \tag{5.13}$$
For a system with \( F \) degrees of freedom, Matrices \([A]\) and \([K]\) are \( N \times F \) matrices. \([K]\) can be obtained by solving the \( N \times N \) system. Influence coefficients relate the secondary velocities with the driving velocities and are very important parameters in the kinematic analysis. Beard used it as \textit{kinematic coefficient} to analyze the suspension system \cite{Beard1993}. Matrix \([K]\) is called \textit{Jacobian matrix} in robotics \cite{Craig1986}. It is a multi-dimensional derivative matrix and acts as a velocity transformation. The Jacobian connects the instantaneous velocity among joints and end actuator, i.e., finger of robotics. It is a time-varying linear transformation matrix. In robotics analysis, the number of degrees of freedom, and number of columns equals the number of joints.

Matrix \([K]\) describes the kinematic characteristic. It is velocity independent and decided by position variables. It is used in a time-varying environment and describes the geometric compliance. It is used in characterization of the motion of the mechanism to perform a mapping from driving velocity to secondary velocity.

\section*{Force}

Differentiating Equation 5.7

\[
\sum_{j=1}^{N} \frac{\partial L_i}{\partial u_j} \delta u_j = - \frac{\partial L_i}{\partial x} \delta x \tag{5.14}
\]

in matrix form, it becomes

\[
[B] \{\delta U\} = -[A] \{\delta X\} \tag{5.15}
\]

Using virtual work principle,

\[
[R] \{\delta U\} = -[D] \{\delta X\} \tag{5.16}
\]

\([D]\) represents driving force and \([R]\) represents force acting on driven bodies of a mechanism. Both are generalized forces. \(\{\delta X\}\) and \(\{\delta Y\}\) are virtual displacements corresponding to forces \([D]\) and \([R]\) respectively.

Using the relationship between \(\{\delta U\}\) and \(\{\delta X\}\) in Equation 5.15,

\[
[R] (-[B]^{-1}[A] \{\delta X\}) = [D] \{\delta X\} \tag{5.17}
\]

This equation is used for all \(\delta x\), therefore
The expression \(-[B]^{-1}[A]\) relates the output force \([R]\) and driving force \([D]\) by virtual work principle. It transforms the output force to the driving force. In the robotics analysis, it connects the force at the finger with the torques at the joints [Craig 1986]. The expression \(-[B]^{-1}[A]\) represents the force transmission characteristic of the mechanism. It is a very important parameter for statics and dynamics analyses.

**Sensitivity and Newton Method**

The Newton - Raphson method uses an iterative procedure to find the zeros of the function, which is expressed as

\[
\{X_{i+1}\} = \{X_i\} - [J]^T_i \{F\}_X
\]  

(5.19)

Here:  
\(\{X_{i+1}\}\): approximate roots after \(i+1\) iterations,  
\(\{X_i\}\): approximate roots after \(i\) iterations,  
\(\{F\}_X\): function values at \(x_i\),  
\([J]^T_i\): first derivative matrix of the function at \(x_i\) [Burden \textit{et al.} 1993],  
i: iteration counter.

In the computation, the determinant of \([J]\) is calculated in the solution procedure. If it sinks below a specified tolerance, the warning is printed out for singularity. The general algorithm is in two steps.

1. \([J]^T_i \{\Delta X\} = -\{F\}_X\)  
   (5.20)

2. \(\{X_{i+1}\} = \{X_i\} + \{\Delta X\}\)  
   (5.21)

For the mechanical assembly, \(F\) function is the assembly function. The variables of the function are grouped as independent and dependent variables, as described in Chapter 2. If the independent variables change from \(\{X\}\) to \(\{X\} + \{dX\}\), the new exact dependent variables \(\{U\}\) can be obtained through the Newton method. The iterative procedure can be expressed explicitly for assembly as:

1. \([J]^T_{X \cdot U_i} \{\Delta U\} = -\{F\}_X \cdot U_i\)  
   (5.22)
2. \( \{U_{i+1}\} = \{U_i\} + \{\Delta U\} \) \hspace{1cm} (5.23)

In terms of the geometric sensitivity matrix \([A]\), \([B]\), and residual, Equation 5.20 becomes

\[ [B]_{X^+,U_i} \{\Delta U\} = -[\text{Res}]_{X^+,U_i} \] \hspace{1cm} (5.24)

The subscripts represent points where the matrices are evaluated.

From the direct linearized method in Chapter 2, Equation 2.19 can be written as

\[ \{\Delta U\} = -[B]_{X,U}^{-1} [A]_{X,U} \{dX\} \] \hspace{1cm} (5.25)

The \(\{\Delta U\}\) from Equation 5.25 are used further in the statistical or worst case analysis of the tolerance. Expression \([A]\{dX\}\) can be expressed in incremental form for residual, provided the variation is small.

\[ [B]_{X,U} \{\Delta U\} = -[\text{Res}]_{X,U} \] \hspace{1cm} (5.26)

Comparison of the Equations 5.24 and 5.26 can be made. If geometric sensitivity matrix \([B]\) is evaluated as \(\{X^+\}\), i.e., \(\{X\} + \{dX\}\), the linearized method is equivalent to one step of the Newton method. As \(\{dX\}\) are small, the \(\{\Delta U\}\) are very close to each other.

**Summary**

From the above analysis, tolerance sensitivity matrix \([S]\) for variation, influence coefficient matrix \([K]\) for kinematics, and force transmission characteristic have a common expression as \(-[B]^{-1}[A]\). They are very important transmission characteristics in corresponding domains. They appear in variation, motion and force analysis as transformation matrices. They can be used to find assembly variation, output velocity and mechanical advantage respectively. All the transformation matrices discussed depend only on the geometry. Geometric sensitivity matrices \([A]\) and \([B]\) have a relationship with the Newton method. The similarity between the solution procedure for the velocity equation and variation equation should be readily apparent. There must exist a certain analogy between the kinematics analysis and tolerance analysis, as \([S]\) and \([K]\) have the same expression. The analogy will be discussed further in Chapter 6.
5.2 SOME QUALITIES OF SENSITIVITY

Coordinate System Dependence of Geometric Sensitivity

The loop functions are scalar algebraic equations for 2D assemblies [Marler 1988]. For a closed loop assembly, the first and the last vectors are connected tail to head to form closed vector loops. For an open loop, the vector representing gap or clearance has its head and tail connected to the last and the first vectors.

In 2-D, each vector loop is represented by three scalar loop equations. If there are n vectors in the loop, L represents length and θ represents the angle relative to the previous vector, and scalar equations are expressed in terms of residuals in the coordinates.

\[
H_X = \sum_{i=1}^{n} L_i \cos \sum_{k=1}^{i} \theta_k
\]

\[
H_Y = \sum_{i=1}^{n} L_i \sin \sum_{k=1}^{i} \theta_k
\]

\[
H_\theta = \sum_{i=1}^{n+1} \theta_k
\]

The geometric sensitivity is composed of the derivatives of the loop functions. The derivative of the loop functions can be expressed as the following.

Geometric sensitivities for the length are

\[
\frac{\partial H_X}{\partial L_i} = \cos \left( \sum_{k=1}^{i} \theta_k \right)
\]

\[
\frac{\partial H_Y}{\partial L_i} = \sin \left( \sum_{k=1}^{i} \theta_k \right)
\]

\[
\frac{\partial H_\theta}{\partial L_i} = 0
\]

Geometric sensitivities for the angle are

\[
\frac{\partial H_X}{\partial \theta_i} = \sum_{j=i}^{n} [- \text{sign } \theta_j] L_j \sin \left( \sum_{k=1}^{j} \theta_k \right)
\]
\[
\frac{\partial H_Y}{\partial \theta_i} = \sum_{j=1}^{n} [\text{sign} \theta_j] L_j \cos \left( \sum_{k=1}^{j} \theta_k \right) \tag{5.34}
\]

\[
\frac{\partial H_\theta}{\partial \theta_i} = [\text{sign} \theta_i]^* 1 \tag{5.35}
\]

The derivatives are grouped in [A] and [B] matrices as shown in Equation 2.18. They are expressed in terms of the established coordinate system for the vector model.

The geometric sensitivity of the length is a unit vector of the corresponding vector. The direction cosines are corresponding values to different coordinate systems. The geometric sensitivity of the angle has a connection with the coordinates of the joint. The geometric sensitivities are not the same if the relative position to the coordinate system changes.

The geometric sensitivity is coordinate system dependent.

**Coordinate System Independence of Tolerance Sensitivity**

The geometric sensitivities are intermediate values for the tolerance sensitivities. They are used to find the tolerance sensitivity as shown in the Equation 5.1.

If the position of the assembly described by vector model changes, it can be represented by the homogeneous transformation. The operation includes translation and rotation. The original geometric sensitivity matrices need to be operated by the transformation matrix [T] so they can be described in the global coordinate.

\[
[A_n] = [T][A] \quad \text{and} \quad [B_n] = [T][B] \tag{5.36}
\]

The new tolerance sensitivity will have the expression

\[
[S_n] = - [B_n]^{-1}[A_n] \tag{5.37}
\]

Because of the matrix identity property, Equation 5.37 gives the same solution as Equation 5.1.

This means that the tolerance sensitivity is coordinate system independent. The tolerance sensitivities have the same values, regardless of what coordinate system is used to model the assembly. Even though different geometric sensitivity matrices [A] and [B] are obtained due to the different relative positions, the tolerance sensitivity is the same.
Determinant of Matrix \([B]\)

Tolerance sensitivity can be represented as

\[
S = -[B]^{-1}[A] = - \frac{[B]^*A}{|B|}
\]  
(5.38)

This is also the common attribute discussed in the last section, which represents the transformation characteristics in the domains of variation, motion and force. \([B]^*\) is the adjoint matrix of matrix \([B]\). Matrix \([B]\) is a first order partial derivative matrix. The inverse of \([B]\) matrix acts as an important element in transformation for error, velocity, and force. For an assembly, the determinant of \([B]\) matrix gives the denominator of the sensitivity matrix, as shown in Equation 5.38. The smaller the determinant is, the more sensitive the assembly will be to manufacturing variations. For a mechanism, if the determinant of \([B]\) is smaller, the mechanical advantage is smaller and the output velocity is larger. This is very important in assembly and mechanism analysis. The factors in the determinant of matrix \([B]\), such as transmission angle and link parameters, have been used in the synthesis of sensitivity for the four-bar mechanism [Lee et al. 1992] [Faik et al. 1991] [Wu et al. 1992].

In the case with three dependent variables, the determinant of matrix \([B]\) can be represented graphically. The value of the determinant is the value of a parallelepiped formed by three derivative vectors, i.e., \(\begin{bmatrix} \frac{\partial H_x}{\partial u_j}, \frac{\partial H_y}{\partial u_j}, \frac{\partial H_\theta}{\partial u_j} \end{bmatrix}^T\). The value is the box product of the vectors, also called scalar triple product. The three derivative vectors are drawn from a common point \(o\).

For the derivatives relative to a length, the third component is always zero. The vector as a unit vector stays in the XY plane. For the derivative relative to an angle, the third component has a magnitude of one.

Figure 5.1 shows the case with three dependent angles. The value of the determinant of \([B]\) equals the volume of a parallelepiped. The components of \(a_i\) in the object shown are the partial derivatives from geometric sensitivity \(B\). Each \(a_i\) vector has a component in \(\theta\) of one.
Figure 5.1 Determinant of matrix [B] (case with three dependent angles)

Figure 5.2 shows the case with two dependent angles and one dependent length. The value of the determinant equals the volume shown. The vector e1 is a unit vector in the XY plane.

Figure 5.2 Determinant of matrix [B] (case with two dependent angles)

Figure 5.3 shows the volume as value of the Determinant of matrix [B] for the case with one angle and two lengths. There are two unit vectors: e1 and e2.

Figure 5.3 Determinant of matrix [B] (case with one dependent angle)
In Figure 5.4, the three dependent variables are three lengths and [B] matrix is singular. The singularity of the [B] matrix causes the spanning derivative vectors to be coplanar. The volume spanned is zero as shown in the plot. One derivative vector is a linear combination of the other two, or three derivative vectors are dependent upon each other. In a mechanism, this means the mechanism is in a singular configuration [Paul 1979].

![Diagram showing three dependent lengths and coplanar vectors](image)

Figure 5.4 Determinant of matrix [B] (case with three dependent lengths)

For a case with more than three dependent variables, expansion by cofactor can break the system down to the cases described above.

**Discussion**

Geometric sensitivity is coordinate system dependent and tolerance sensitivity is coordinate system independent. It is just like looking at an object from a different view. The assembly or geometry of the object does not change with the viewpoint and only the geometric sensitivity representation changes. The representation is relative and the relationship is absolute. We are more interested in the relationship itself than the representation, i.e., interested in the tolerance sensitivity rather than the geometric sensitivity.

In reality, the sign and the magnitude in the sensitivity have definite physical meanings referring to the variations. The sign gives the direction effect, and the absolute value gives the magnitude effect. They are key factors to reveal the nature of the sensitivity. The effect of both shall be considered in the variation propagation process. Consideration without the sign or direction makes the estimation more conservative. The potential for improving the quality of the assembly may be ignored. The tolerance sensitivity needs to be further analyzed.
5.3 VARIATION POLYGON

Assembly tolerance represents the variation relationship between dependent and independent variables. From intuition, the tolerance sensitivity should be coordinate system independent. In the vector space, the closed loop relationship can be established from any point in the loop. The equations always result in zero lengths and constant angles. The tolerance sensitivity may be self-explanatory, as if you were sitting on an object in space in order to see the variation relationship in the object itself.

If tolerance sensitivity can be represented as an algebraic relationship instead of numerical data, the nature of tolerance sensitivity can be better understood. Tolerance analysis can be performed at the early design stage. It can help nominal design as well as tolerance design. There is a need to find a way to describe the relationship absolutely and represent tolerance sensitivity directly.

The variation and the assembly closure are two important elements in the analysis. This section introduces a new approach to analyzing the tolerance sensitivity, which is called the variation polygon. The approach is geometric and analytic. The representation for tolerance sensitivity differs from other methods in that it is more in-depth. An example problem of a one-way clutch assembly presented in Chapter 4 is used to illustrate the concept.

Figure 4.10 shows the vector loop and the variables in a one-way clutch assembly. Each dimension is represented by a vector with a specified nominal length and relative angle to the previous vector. It has three independent variables: a, c and e. All of them are specified in the design and the component tolerances are given. There are three dependent variables: b, φ1 and φ2. Their nominal values can be determined by solving the loop equations or by a careful graphical layout. The variations of dependent variables cannot be specified directly in the design and are only decided by the assembly resultant. The tolerance sensitivities will be:

\[
[S] = \begin{bmatrix}
\frac{\partial b}{\partial a} & \frac{\partial b}{\partial c} & \frac{\partial b}{\partial e} \\
\frac{\partial \phi_1}{\partial a} & \frac{\partial \phi_1}{\partial c} & \frac{\partial \phi_1}{\partial e} \\
\frac{\partial \phi_2}{\partial a} & \frac{\partial \phi_2}{\partial c} & \frac{\partial \phi_2}{\partial e}
\end{bmatrix}
\]  

(5.39)
This is a derivative matrix. It can be obtained from the procedure described in Chapter 2. It involves the derivatives of constraint functions for geometric sensitivity and matrix calculations. The geometric sensitivity matrices [A] and [B] and tolerance sensitivity matrix [S] for clutch are included in the Appendix.

**Key Concept: Kinematic Adjustment**

![Diagram](image)

Figure 5.5 Closure error $\Delta H$ due to variation of the independent variables
The new approach is based on a geometric analysis. In a mechanical assembly, any variation of independent variables due to manufacturing makes the vector loop fail to close, if there are no kinematic adjustments. This can be seen in Figure 5.5. For clarity, the variations are greatly exaggerated. The densely dotted vector ΔH represents the closure error produced by variations \( \Delta a \), \( \Delta c \) and \( \Delta \phi \). It produces non-zero residuals in the loop equations. The closure error of the loop must be adjusted by dependent variables \( b \), \( \phi_1 \) and \( \phi_2 \). The adjustments are shown as two boldly dotted vectors. Graphically, this illustrates the kinematic adjustment process, which brings the loop closed again. Variation of the dependent variables may be described in this manner.

As the tolerance sensitivity is evaluated at the nominal position, the most critical parameters can be brought together to be analyzed. They are the variations or the changes of the nominal in the design. By the kinematic adjustment, the changes of the closure condition produced by the variations of the independent variables are compensated for by the variations of the dependent variables, if the assembly still assembles. That means the system closure condition needs to be satisfied and the sum of the variation vectors must add to zero. Thus, a new vector approach has been created where the relative effects of each independent vector variation on the dependent vector variation can be seen visually and calculated from the vector geometry. The resulting polygon is called the variation polygon. It provides both geometric and analytic information.

The variation polygon is analogous to the velocity polygon in kinematics. It is a graphical solution procedure that gives powerful insight into the interaction of dependent and independent vector variables. It yields much more information than examining the numerical value in the sensitivity matrix, because it is more relationship-oriented, rather than simply data-oriented.

The information about the degree of sensitivity of the dependent variables to the independent variables is provided by tolerance sensitivities. It helps the designer to allocate the tolerance for the independent variables and find the variation of the dependent variables at the nominal dimensions. The variation polygon allows tolerance sensitivities to be found analytically, directly from the geometry, without involving extensive matrix calculations.

**Variation Polygon Example**

The variation polygon is a closed polygon. It is composed of vectors representing the variations of the independent and dependent (assembly) variables.
In the clutch assembly example, variation vectors $\Delta \overline{a}$, $\Delta c$ and $\Delta \overline{e}$ are added vectorially in the variation plane, as shown in Figure 5.6. They represent the variations or the changes of the nominal of the independent variables in the design. The corresponding kinematic adjustments needed for the closure condition are shown as dashed vectors. In the clutch, the variation of the dependent variable $b$, i.e., $\Delta b$, is a horizontal vector. The variations of the angle $\phi_1$ and $\phi_2$ rotate together to produce the result that is represented as $R\Delta \phi_1$ to contribute to kinematic adjustment. Two dashed lines determines the magnitude and direction of $\Delta b$ and $R\Delta \phi_1$. Thus, the dependent or assembly variations required to produce closure may be determined directly from the variation polygon, without resorting to linearized equations and matrix algebra.

![Figure 5.6 Variation polygon for one-way clutch](image)

**Tolerance Sensitivity from Variation Polygon**

The tolerance sensitivity can also be derived directly from the variation polygon, if independent variables are varied one at a time and the orientations of the vector variations of the dependent variables are held the same as in Figure 5.6.

**Variation due to the Change in Dimension $a$**

Given $\Delta \overline{a}$, then, the variation polygon is as shown in Figure 5.7.
Figure 5.7 Variation polygon for first column of tolerance sensitivity matrix [S]

All the relationships of variations of the dependent variable $u_1$ to the variation of the independent variable $\frac{\bar{a}}{2}$, i.e., $\frac{\Delta u_1}{\Delta \frac{\bar{a}}{2}}$, can be obtained from this plot.

Look at the variation $\Delta b$ due to the variation $\Delta \frac{\bar{a}}{2}$ from the variation polygon; the magnitude of the variation is $\frac{1}{2 \tan \phi_1}$, and the direction of the variation of $b$ is opposite of the original $b$ vector. This means that the $b$ is decreased when the variation of $\frac{\bar{a}}{2}$ is in a positive direction. In other words, an increment of $\frac{\bar{a}}{2}$ makes $b$ decrease. Therefore, the negative sign shall be attached for this variation to show the negative effect for $b$.

Now, look at the variation $\Delta \phi_1$ due to the variation $\Delta \frac{\bar{a}}{2}$ from the variation polygon; the magnitude of the variation is $\frac{1}{R \sin \phi_1}$. From the direction of the variation polygon, the $\Delta \phi_1$ makes $\phi_1$ decrease and the negative sign is attached for this variation to show the negative effect for $\phi_1$. The variation polygon provides the information of the magnitude and the direction of the variation for the dependent variables due to the variation of the corresponding independent variable.

Deriving from the geometry of the variation polygon, the sensitivities due the variation $\Delta a$ are:

$$\frac{\Delta b}{\Delta a} = -\frac{1}{2 \tan \phi_1} = -\frac{1}{2 \cot \phi_1}$$  \hspace{1cm} (5.40)

$$\frac{\Delta \phi_1}{\Delta a} = -\frac{1}{2R \sin \phi_1} = -\frac{1}{2R \csc \phi_1}$$  \hspace{1cm} (5.41)
\[ \frac{\Delta \phi_2}{\Delta a} = \frac{\Delta \phi_1}{\Delta a} = \frac{1}{2R \sin \phi_1} = \frac{1}{2R \csc \phi_1} \]  

(5.42)

**Variation due to the Change in Dimension c**

Given \( \Delta c \), then, the variation polygon is shown in Figure 5.8:

![Variation polygon for second column of tolerance sensitivity matrix [S]](image)

**Figure 5.8**  Variation polygon for second column of tolerance sensitivity matrix [S]

The relationship can be derived from the variation polygon in Figure 5.8:

\[ \frac{\Delta b - \Delta c \sin \phi_1}{\Delta c \left(1 + \cos \phi_1\right)} = \frac{1}{\tan \phi_1} \]  

(5.43)

After simplifying and considering direction, the sensitivities relative to \( c \) are:

\[ \frac{\Delta b}{\Delta c} = \frac{1 + \cos \phi_1}{\sin \phi_1} \]  

(5.44)

\[ \frac{\Delta \phi_1}{\Delta c} = \frac{1 + \cos \phi_1}{R \sin \phi_1} \]  

(5.45)

\[ \frac{\Delta \phi_2}{\Delta c} = \frac{\Delta \phi_1}{\Delta c} = \frac{1 + \cos \phi_1}{R \sin \phi_1} \]  

(5.46)

That means, an increment of \( c \) makes \( b \) and \( \phi_1 \) as well as \( \phi_2 \) decrease.

**Variation due to the Change in Dimension e**

Given \( \Delta \frac{e}{2} \), then, the variation polygon is shown in Figure 5.9:
The relationships are seen from the above variation polygon. In terms of dimension ε, the sensitivities relative to ε are:

\[
\frac{\Delta b}{\Delta \varepsilon} = \frac{1}{2 \sin \phi_1} = \frac{1}{2 \csc \phi_1} \quad (5.47)
\]

\[
\frac{\Delta \phi_1}{\Delta \varepsilon} = \frac{1}{2R \tan \phi_1} = \frac{1}{2R} \cot \phi_1 \quad (5.48)
\]

\[
\frac{\Delta \phi_2}{\Delta \varepsilon} = \frac{\Delta \phi_1}{\Delta \varepsilon} = \frac{1}{2R \tan \phi_1} = \frac{1}{2R} \cot \phi_1 \quad (5.49)
\]

In the foregoing, all the tolerance sensitivities have been derived from the variation polygon analytically and exactly. The variations of the dependent variables due to the variation of the one independent variable can be derived from one variation polygon for each column of the tolerance sensitivity matrix [S]. It gives the magnitude and direction as well. The derivations from the variation polygon yield explicit expressions for the tolerance sensitivity and variations in concise algebraic form. The information is not only qualitative, but quantitative as well. The further application of the variation polygon in variation for three different cases and nominal analysis will be presented in Chapter 6.

**Implications**

The variation polygon method gives tolerance sensitivity not only an exact analytical, but also a visually graphical representation. It relates the tolerance sensitivity and assembly variation directly to the geometry. The kinematic adjustments to each variation introduced can be seen. The implicit relationships in the assembly loop equations become explicit in the variation polygon method. Extensive calculations and computation errors are avoided. This will simplify the process for tolerance analysis extensively. This
method makes tolerance analysis relationship-oriented rather than data-oriented. The nature of kinematic adjustment is very clearly monitored through the variation polygon. This reveals the relationship more clearly and visually than does the data calculation procedure. It gives much information in a form that is useful in the early design stage.

As a design strategy, tolerance analysis is the micro analysis of the design dimensions and it is performed at a specified nominal design point. A different nominal design belongs to macro consideration in design, and correct nominal designs need to be carefully chosen. The variation polygon method can give fast feedback about the consequences of manufacturing variations and provides a way to analyze the assembly variations at any nominal design. This can accommodate changes in the nominal dimensions. It can give tolerance information before the final dimensions are decided. It can facilitate the interactive parallel design of the nominal and tolerance levels. It is a convenient evaluation tool for the early design stage.

5.4 SUMMARY

This chapter analyzed three different domains: variation, motion and force; and pointed out their common attributes. There exists an analogy between the tolerance analysis and kinematic velocity analysis. The coordinate system dependence of the geometric sensitivity and coordinate system independence of the tolerance sensitivity have been discussed. A way to analyze tolerance sensitivity to reveal this nature is possible. A new way to analyze the tolerance sensitivity is developed through a one-way clutch example. The interrelationship between independent variables and dependent variables becomes crystal-clear. The new method provides dynamic information about the kinematic adjustment. It is a very promising approach. Further analysis from geometric and analytic approach for different assembly cases will be presented in Chapter 7.
Chapter 6

IMPLICATIONS OF THE VARIATION POLYGON

The variation polygon method has been developed in Chapter 5. The information from the variation polygon can be used further. This chapter presents the application of variation polygon to tolerance accumulation and analyzes the analogy between velocity analysis in mechanism kinematics and variation analysis in assembly tolerances.

6.1 APPLICATION TO TOLERANCE ACCUMULATION

Tolerance sensitivity can be derived from the variation polygon as discussed in Chapter 5. The variation polygon presents tolerance sensitivity, as well as the variations of dependent variables due to the variations of the independent variables. All the variations of dependent variables are available here for further analysis. The dependent variations can be accumulated by three analyses, i.e., worst case, statistical analysis and a specific case.

The worst case and statistical analysis are used extensively for predicting tolerance accumulation in assemblies as shown in Equation 6.1. Worst case uses the sum of absolute values, and statistical analysis uses RSS (root sum squares) of the dependent variations due to each manufacturing variation. These two cases are used to estimate the assembly variations for the entire population. Component variations have positive values and dependent variations by each manufacturing error are not allowed to cancel each other in the accumulation.

\[
\{dU\} = \sum \left| \frac{\partial u_i}{\partial x_i} \right| dx_i \quad \{dU\} = \sqrt{\sum \left( \frac{\partial u_i}{\partial x_i} dx_i \right)^2} \tag{6.1}
\]

Equation 6.2 corresponds to a specific case of an assembly, which was not analyzed before. The resultant variation dU equals a linear combination of the variations produced by each manufacturing error. The effects of the signs in tolerance sensitivity matrix are retained. It describes variations for an assembly with specified dimensions and specific manufacturing error dX. It represents selecting a single set of parts and assembling them to determine the resultant variations of the assembly features.

\[
\{dU\} = \sum \frac{\partial u_i}{\partial x_i} dx_i \tag{6.2}
\]
Returning to the one-way clutch assembly as an example, let manufacturing variations, $\Delta a$, $\Delta c$ and $\Delta e$, have positive magnitudes and directions as shown in Figures 5.7 to 5.9. By the variation polygon, the variations $\Delta b$ can be determined for each independent variation. They are parallel, but different in directions. The total variation in $b$ may then be estimated by adding the three increments of $b$, using worst, statistical and specific case analysis. Any of the three representations of the variation becomes very convenient by using the results from the variation polygon, and variation vectors can be shared among different representations. Three methods of adding the $\Delta b$ are shown in the following plots.

For the worst case representation, the variation vectors are added in their absolute value forms without considering directions. The worst case estimate of $\Delta b$ is shown in Figure 6.1.

**Figure 6.1** Representation for variation of the worst case

**Figure 6.2** Representation for variation of the statistical analysis
For the RSS, i.e., statistical analysis representation, the variations must be added orthogonally. The RSS estimate of the magnitude of $\Delta b$ is shown in Figure 6.2. The magnitude is much smaller than the worst case. Note that the resultant $\Delta b$ must be rotated to the horizontal position, the same as the nominal $b$ vector.

The direction of each vector in the variation polygon describes an increase or decrease in the variables. If the $\Delta b$ vector is in the opposite direction of the nominal $b$ vector, the corresponding tolerance sensitivity is negative. This phenomenon shows the connection between the algebraic expressions and the geometric representations. In a specific assembly case, when the variation $\Delta b$ by each component variation are summed by polygon approach, the effect of direction and magnitude can be considered in the propagation process of variations. The accumulation process of variations in the assembly for a specific case is clearly represented graphically in Figure 6.3. In this figure, the variation vectors are added together. This means that the signs of the sensitivity and component variation must be retained. Because of possible differences in the signs in the tolerance sensitivities as well as the manufacturing errors of the components, the resultant contribution to $\Delta b$ by each component error in the assembly may cancel each other to some extent. The magnitude is much smaller than both worst case and statistical analysis. However, it represents a specific assembly case instead of the entire population.

![Figure 6.3 Representation for variation of a specific assembly](image)

The variation polygon has all the information for the tolerance sensitivity and variations, which are required for the worst, statistical and specific case analysis. Each of these analyses uses different ways to stack the variations produced by individual manufacturing variations. This stackup process can be simulated and monitored. It allows the designer to see the accumulation of the variations, and the effect of tolerance allocation can be "observed" directly by the information from the variation polygons.
6.2 BENEFIT OF VARIATION POLYGON

Choosing Nominal at an Early Stage

Changes in the nominal design dimensions are most likely to occur at an early design stage. Tolerance analysis in this stage needs to accommodate this change. Tolerance analysis in this stage will be made possible and convenient by using the variation polygon. The robust region of design space may be determined by examining the variation polygon and estimating the effect of a change in the nominal dimensions. The geometry may reveal which direction to change the nominal dimensions for a more robust assembly. This means that the variation of the dependent variables due to the same variation of the independent variables may be much less.

Let’s continue to use the clutch as an example. If another set of nominal values of independent variables a, c and e are chosen, the R and φ would change correspondingly. However, the relationship in the variation polygon is still the same from the polygon method. The change of the tolerance sensitivities can be seen qualitatively and quantitatively from the changed variation polygon. This would effectively provide information about the preferred range of nominal values for tolerance sensitivity.

![Variation Polygon Diagram]

Figure 6.4 Suggestion of the nominal determination

Figure 6.4 shows two different variation polygons for the clutch due to a change in e. Comparing the two plots, it can be seen that when the nominal angle φ₁ is smaller, the variation of b and φ₁ due to the same variation in e will be much larger. Considering
variations alone, if it is allowable from other requirements, the angle $\phi_1$ should be as large as possible. If the objective is to minimize the variation of the angle $\phi_1$, the optimization algorithm will always pick the biggest allowable value for the nominal angle $\phi_1$. This conclusion can also be seen directly from the contour plot of the $d\phi_1$, Figure 4.12. However, the variation polygon will give the same information from nominal considerations with much less effort. It is therefore a very convenient way to find tolerance sensitivity and variation information in the early tolerance design stages.

The variation polygon provides the relationships, which can be described graphically or analytically. The relationships hold for all the different nominal dimensions. The design alternatives due to different nominal dimensions can be easily analyzed for assembly tolerance. Since the tolerance sensitivities and variations can be described as a geometric relationship rather than data for specified dimensions, the variation polygon is able to provide suggested nominal dimensions for less variation of the dependent variables.

**Information from Variation Polygon**

The variation polygon provides valuable information for tolerance analysis.

--- the relationship between the variations of the independent variables and dependent variables in the assembly are preserved;

--- the tolerance sensitivity matrix in analytical form is obtained for one column of the tolerance sensitivity matrix from each variation polygon;

--- the variations of dependent variables in each variation polygon can be used very easily for the worst case and statistical analysis. They only involve absolute and orthogonal addition of the variation vectors;

--- the specific analysis case for dependent variations due to manufacturing errors of all independent variables can be obtained. They only involve the summation of variation vectors and include effect of directions and magnitudes.

--- the nominal value for the robust design space in the early design stage may be visually suggested, which will be a great help for performing preliminary tolerance analysis in design;

--- all the above information at any nominal value can be obtained by polygon approach.

The advantages of the variation polygon are:

--- giving a visual geometric relationship about tolerance sensitivity and variation;
--- avoiding extensive computation, possible truncation and round off error from the numerical approach;
--- giving accurate tolerance information in the early design stage for the designer, which is very effort-extensive and prohibitive otherwise;
--- facilitating tolerance assignment;
--- giving the designer direct feedback about:
  the nominal selection;
  the assignment of tolerances of the independent variables;
  kinematic adjustments.

From the analysis of the variation polygons, a designer can gain more insight about assembly tolerances. Variation polygon reveals more about the nature of tolerance sensitivity, with far-reaching implications. Combining the advantages in the analytical and graphic analysis approach makes the analysis more relationship informative. It gives direct information of the tolerance sensitivities and assembly variations with direction and magnitude, instead of through the derivative matrix calculation. It will speed the interactive feedback process in the design stage. The analysis is fast, accurate, direct and relationship-oriented. It has great potential to facilitate tolerance analysis at the early design stage.

6.3 ANALOGY BETWEEN KINEMATIC VELOCITY AND ASSEMBLY VARIATION

Velocity analysis examines the nominal mechanism and describes its motion as a function of time. Dimensions are constant. Variation analysis examines all the assemblies at once and describes the variations of each assembly from the nominal. Dimensions vary from assembly to assembly.

There is a common attribute in variation and motion analysis, as discussed in Chapter 5. However, no work has been found to analyze the analogy between these two fields. There is a connection between variation analysis in assembly tolerance and velocity analysis in mechanism kinematics. This section will illustrate the analogy through analysis of a 3-bar truss example. The structure is analyzed to obtain the velocity polygons and variation polygons, and the results are compared.

Assembly and Mechanism

The structural assembly is composed of three members, joined by pins, forming a triangular truss. The inaccuracy in the length of each member produces variations in the
assembly. The dependent angles between mating pairs will vary. The analysis of this assembly is a tolerance problem. This assembly can be schematically represented, as shown in Figure 6.5.

![Figure 6.5 Schematic diagram for truss](image)

If the variations of the lengths are considered one at a time, the schematic diagram shows how the length variation produces a corresponding change in the three dependent angles, as shown Figure 6.6. This represents a simplified truss assembly.

![Figure 6.6 Angular variations introduced by a single length variation](image)

If the length change is considered to be time-varying, the above figure can represent a slider-crank mechanism. The variation of the length corresponds to a linear velocity, and the angular variations correspond to angular velocities. The input is the movement of the slider. The resulting mechanism is shown in Figure 6.7.

![Figure 6.7 Slider-crank mechanism](image)

The slider velocity is accommodated by the angular velocities at the joints. This means given one velocity input, the angular velocities can be solved. This describes a kinematic problem.
**Velocity Analysis for a Mechanism**

A graphical method is used to analyze the relationship between linear velocity $\frac{dL_1}{dt}$ and angular velocities $\omega_i$ ($i = 1, 2, 3$). In the following, the kinematic inversions of the crank slider mechanism are created and examined. By making a different link in a kinematic chain become the fixed member, three different mechanisms are obtained. Inversions do not affect the relative motion between links. However, the absolute motion of the points moving with a particular link will be different for each fixed link [Sun et al. 1978].

In the first kinematic inversion, link L3 is fixed. The velocity polygon is shown in Figure 6.8.

![Figure 6.8 Inversion and graphical analysis for fixed L3](image)

The magnitudes and directions of velocities are obtained from rigid body analysis. The angular velocity $\omega_1$ and $\omega_3$ can be solved if the movement of the slider is given. The angular velocity $\omega_2$, which is not shown explicitly in the velocity polygon, can be derived from the Equation 6.3.

$$\omega_2 = - (\omega_1 + \omega_3)$$  \hspace{1cm} (6.3)

In the second inversion, link L1 is fixed. It is a basic slider-crank mechanism used in automotive engines. The kinematic inversion and the velocity polygon are shown in Figure 6.9.
In the third inversion, link L2 is fixed. It becomes an oscillating cylinder mechanism. The kinematic inversion and the velocity polygon are shown in Figure 6.10.

The relationship between the angular velocities and the linear input velocity in the mechanism can be solved from the velocity polygons. Similar relationships between angular velocities and linear velocities, e.g., \( \frac{dL_2}{dt} \) or \( \frac{dL_3}{dt} \), can be derived by symmetry properties.

**Variation Analysis for Assemblies**

The analogy between the velocity analysis and variation analysis can be seen by comparison of the velocity polygon of the mechanism and variation polygon of the assembly.

The velocity polygon, for the inversion case of fixed L1, as shown in Figure 6.9, is used as an example to develop the variation polygon. The process is shown in Figure 6.11.
Implications of the Variation Polygon

First, the absolute velocity is explicitly expressed as relative to the grounded point. Second, the relative relationship is reversed in the expression and direction in graph. Thus, the velocity of the grounded point is represented by a velocity polygon.

Then, if the time-varying factor is taken out, i.e., multiplying each velocity by dt, the velocity polygon becomes a polygon representing the relationship of variations. It can represent the kinematic adjustment due to the change of the length. The dt is decided by Equation 6.4.

\[ dt = \frac{dL_1}{V_{B2/B1}} \]  \hspace{1cm} (6.4)

From velocity polygon to variation polygon

From variation polygon to velocity polygon

Figure 6.11 Developments of polygons

The variation of \( \phi_1 \) and \( \phi_2 \) can be directly derived from the variation polygon. At last, \( d\phi_3 \) can be obtained by Equation 6.5.

\[ d\phi_3 = - (d\phi_1 + d\phi_2) \]  \hspace{1cm} (6.5)

The variation polygon in Figure 6.11 is equivalent to the variation polygon derived from the truss assembly with the starting point at B. If only L1 changes, the variation polygons can be derived from the velocity polygons of different kinematic inversions of the slider-crank. The three inversions for fixed L1, L2, and L3 in the mechanism correspond to truss assemblies with their starting points at B, C, and A, respectively.

Figure 6.12 shows all the variation polygons, if the change of L1 remains the same. Corresponding velocity polygons are included in the figure also. There is a significant analogy between the velocity polygon and the variation polygon. In the velocity polygon, the vectors represent the velocity of the output variables of the mechanism for a single input velocity. In variation polygon, the vectors represent the dependent variations for a single
manufacturing variation. Care must be taken for the directions of the vectors in the derivations.

\[ \text{dL1} \]
\[ \text{L1}\cdot \text{d}\phi_1 \]
\[ \text{L2}\cdot \text{d}\phi_3 \]
\[ \text{L3}\cdot \text{d}\phi_2 \]

a) variation polygons for a simplified truss assembly

\[ \text{VB2/B1} \]
\[ \text{VB1} \]
\[ \text{VB2} \]

\[ \text{VC/B1} \]
\[ \text{VC} \]
\[ \text{VC/2} \]

\[ \text{VA/B2} \]
\[ \text{VB1/A} \]

b) velocity polygons for a slider-crank mechanism

Figure 6.12 Corresponding variation and velocity polygons for three inversions

There is one dependent angle that is implicitly expressed by the variation polygon in each inversion: \( \text{d}\phi_2 \) in the inversion of fixed \( L3 \), \( \text{d}\phi_3 \) in the inversion of fixed \( L1 \), and \( \text{d}\phi_1 \) in the inversion of fixed \( L2 \). It is of importance to note that inversion of a mechanism in no way changes the relative motion between the links [Martin 1969]. For example, if link \( L3 \) rotates clockwise a certain angle relative to link \( L1 \), the slider will move to the right a definite amount along a straight line on link \( L1 \), no matter what link is held fixed. Therefore, the relative relationships among the variations are the same for all the variation polygons. This means that difference in the starting points in derivation does not change the relative relationship between the variations.

Following the procedure described in Chapter 5, the first column of the tolerance sensitivity matrix, as well as the velocity ratio of angular velocities \( \omega \) to input velocity \( \overset{\cdot}{L}_1 \), i.e., \( \frac{\text{dL1}}{\text{dt}} \), can be obtained from polygons in figure 6.11. The results are shown in Equation 6.6. Both variation and velocity polygons give the same expressions based on the nominal geometry. The derivations of polygons from each other and the resultant expressions show that there is a strong analogy in the velocity analysis in mechanism kinematics and variation analysis in assembly tolerances.
\[ \begin{bmatrix} \frac{d\phi_1}{dL_1} \\ \frac{d\phi_2}{dL_1} \\ \frac{d\phi_3}{dL_1} \end{bmatrix} = \begin{bmatrix} \frac{\omega_1}{L_1} \\ \frac{\omega_1}{L_1} \\ \frac{\omega_1}{L_1} \end{bmatrix} = \begin{bmatrix} -\frac{\cos\phi_2}{L_3\sin\phi_3} \\ \frac{\cos\phi_1}{L_2\sin\phi_3} \\ \frac{\cos\phi_2 + \cos\phi_1}{L_3\sin\phi_3 + L_2\sin\phi_3} \end{bmatrix} \tag{6.6} \]

It is interesting to see that the relative relationship among the manufacturing variations and assembly variations, and among linear velocity $\frac{dL_1}{dt}$ and angular velocities $\omega_i$ $(i = 1, 2, 3)$, can be obtained from three inversions. Even though the algebraic form looks different, the velocity relationships from different inversions are the same. Chapter 7 will show one consistent representation by using developed pseudovectors.

Following a similar procedure and using symmetry, all the variation polygons for the truss assembly, due to the change of the other two lengths, L2 and L3, can be developed from velocity polygons. The relationship among all linear velocities and the angular velocities can be solved. The other two columns of tolerance sensitivity matrix can be obtained.

Reversibly, for the development from variation polygon to velocity polygon, each vector in the variation polygon needs to be divided by $dt$. It is shown in Figure 6.11 also. The velocity polygon corresponding to different kinematic inversions can be developed. If linear velocities of L2 and L3 are added, three linear velocities are inputs. Such a mechanism would be a multi-input mechanism of three sliders, having three linear velocities $\frac{dL_i}{dt} (i = 1, 2, 3)$ input simultaneously. Then all the relative relationships in the velocity polygon must be carefully decided. The variation polygon has not only tolerance information, but also kinematic velocity information as well. Through this two-way development between the velocity polygon of simple slider-crank mechanism and the variation polygon of the truss assembly, the analogy is more clear.

From the proceeding analysis, the variation polygon provides the tolerance sensitivity and assembly variation in a systematic way. The direct derivation of the tolerance sensitivity and the extension of the method to mechanism analysis look encouraging. The analysis results in both fields will benefit each other. The analogy between kinematics and tolerance broadens the tolerance research in a potential way. As a well-developed field,
kinematic analysis will bring more understanding to tolerance analysis. More light will appear in the future of tolerance analysis.

**Comparison Summary**

Through the truss example, the velocity analysis and variation analysis are compared and listed in Table 6.1. The strong analogy can be found through the table.

**Table 6.1 Comparison summary (part 1)**

<table>
<thead>
<tr>
<th>Analysis</th>
<th>Velocity analysis in mechanism kinematics</th>
<th>Variation analysis in assembly tolerances</th>
</tr>
</thead>
<tbody>
<tr>
<td>Domain</td>
<td>time domain</td>
<td>variation domain</td>
</tr>
<tr>
<td>Simulation</td>
<td>motion of one mechanism at a time</td>
<td>change from one assembly to another (i.e., nominal)</td>
</tr>
<tr>
<td>Vector loop</td>
<td>represent a mechanism.</td>
<td>represent an assembly.</td>
</tr>
<tr>
<td></td>
<td>chain can be open or closed</td>
<td></td>
</tr>
<tr>
<td>DOF</td>
<td>single-input, multi-output at interested points</td>
<td>multi-input, multi-output</td>
</tr>
<tr>
<td>Time</td>
<td>time-varying</td>
<td>time-independent</td>
</tr>
<tr>
<td>Dimension</td>
<td>do not change</td>
<td>subject to manufacturing variations</td>
</tr>
<tr>
<td>Interested in</td>
<td>velocities</td>
<td>variations</td>
</tr>
<tr>
<td>Input</td>
<td>instantaneous velocity</td>
<td>small variations due to manufacturing</td>
</tr>
<tr>
<td>Output</td>
<td>velocities throughout the mechanism</td>
<td>resulting variations throughout the assembly</td>
</tr>
<tr>
<td>Polygon</td>
<td>velocity polygon</td>
<td>variation polygon</td>
</tr>
<tr>
<td></td>
<td>represent relationship among absolute and relative velocities</td>
<td>represent relationship among independent and dependent variations</td>
</tr>
<tr>
<td></td>
<td>specify input velocity, solve for unknown velocity</td>
<td>specify manufacturing variations, solve for unknown dependent variations</td>
</tr>
</tbody>
</table>
### Table 6.1 Comparison summary (part 2)

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>closed vector loop relates absolute and relative velocities</td>
<td>closed vector loop represents kinematic adjustments,</td>
</tr>
<tr>
<td>open vector loop represents absolute velocity of a point</td>
<td>open vector loop represents assembly resultant</td>
</tr>
<tr>
<td>Inversion</td>
<td>start from different joints to get various variation polygons</td>
</tr>
<tr>
<td>ground different links to produce different kinematic inversions</td>
<td>the variation relationship does not change due to different starting points</td>
</tr>
<tr>
<td>the relative motion between links does not change due to different kinematic inversions</td>
<td></td>
</tr>
<tr>
<td>State in Development</td>
<td>well developed field</td>
</tr>
<tr>
<td></td>
<td>new field</td>
</tr>
</tbody>
</table>

### 6.3 SUMMARY

This chapter presented further implications of the variation polygon. Three important analyses, i.e., worst, statistical and specific analysis cases, of the dependent variations can be represented by the information from the variation polygons. The variation polygon is a convenient method to obtain relationships between the variations. The information described from the variation polygon has been listed. The analogy between mechanism velocity and assembly variation has been revealed. This points out a very promising approach to connect tolerance analysis for assemblies to the very well developed field: kinematic analysis for mechanisms. More applications will be discussed in the next chapter.
Chapter 7

VECTOR VARIATION GEOMETRY - CASE STUDIES

Case studies from the newly developed variation polygon and frame polygon approaches are presented in this chapter. They demonstrate the geometric correspondence of tolerance sensitivity to the assembly configuration. Typical cases include static assemblies and mechanisms, single loop cases and multi-loop cases. In order to learn the broad nature of tolerance sensitivity, cases representing different conditions for various dependent lengths and angles are analyzed. The new concept of pseudovectors is defined to construct the frame polygon. The tolerance sensitivity can be defined from assembly geometry, and relationships are more clear than numerical method. The analytical geometric analysis and the derived tolerance sensitivity are presented for the first time. The geometric features that are important to the tolerance analysis are identified, summarized and defined in the process. The results are compared with the available results by other methods. The data verifications are included in the Appendix.

7.1 EXAMPLE 1: BICYCLE CRANK

Problem Description

The bicycle crank assembly in Figure 7.1 is used to secure the pedal crank to the sprocket shaft.

![Figure 7.1 Exploded view of a bicycle crank assembly.](image)

The bicycle crank assembly is composed of three parts: a crank, pin and shaft. The tapered pin holds the crank and shaft together and bears against a mechanical flat on the shaft to transmit torque. The thread part of the pin should be properly extended out of the crank surface to allow locking the pin with a nut.
Figure 7.2 Vector model and network graph for bicycle crank

This assembly can be modeled by one vector loop as shown in Figure 7.2. The dimension i between the shoulder of the pin and surface of the crank is the critical assembly dimension. It must be of sufficient magnitude to allow tightening of the nut without the seating against the pin shoulder. The chain of vectors describes the stacking of dimensions that contribute to tolerance accumulation in dimension i.

Table 7.1 Manufactured dimensions for bicycle crank

<table>
<thead>
<tr>
<th>Variable Name</th>
<th>Description</th>
<th>Basic Size</th>
<th>Tolerance (±)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>Radius of pin hole</td>
<td>4.76 mm</td>
<td>.0075 mm</td>
</tr>
<tr>
<td>b</td>
<td>Distance between centers</td>
<td>7.650 mm</td>
<td>.076 mm</td>
</tr>
<tr>
<td>c</td>
<td>Radius of crank</td>
<td>13.550 mm</td>
<td>.127 mm</td>
</tr>
<tr>
<td>d</td>
<td>Radius of shaft hole</td>
<td>7.85 mm</td>
<td>.0125 mm</td>
</tr>
<tr>
<td>e</td>
<td>Diameter of shaft</td>
<td>15.660 mm</td>
<td>.013 mm</td>
</tr>
<tr>
<td>f</td>
<td>Shaft flat depth</td>
<td>4.500 mm</td>
<td>.050 mm</td>
</tr>
<tr>
<td>h</td>
<td>Pin narrow end width</td>
<td>8.500 mm</td>
<td>.050 mm</td>
</tr>
<tr>
<td>θ</td>
<td>Pin Bevel Angle</td>
<td>86.0°</td>
<td>.5°</td>
</tr>
</tbody>
</table>
Eight independent variables, three dependent variables and associated variations are listed in Table 7.1 and Table 7.2, where the dependent variations are obtained by the linearized method.

Table 7.2 Assembly dimensions for bicycle crank

<table>
<thead>
<tr>
<th>Variable Name</th>
<th>Description</th>
<th>Basic Size</th>
<th>Variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>g</td>
<td>Dependent length</td>
<td>8.71694 mm</td>
<td>1.839058</td>
</tr>
<tr>
<td>i</td>
<td>Contact of pin shoulder</td>
<td>5.085188 mm</td>
<td>1.866243</td>
</tr>
<tr>
<td>ϕ</td>
<td>Dependent angle</td>
<td>94.000 °</td>
<td>0.008727</td>
</tr>
</tbody>
</table>

**Variation Polygon and Derivation**

**Variation Polygon**

For the bicycle crank, the tolerance sensitivity [S] is a 3×8 matrix. It can be obtained from the linearized method as discussed in Chapter 2. The matrix calculation involves the geometric sensitivity matrix [A] with a dimension of 3×8, geometric sensitivity matrix [B] with a dimension of 3×3, and computation of tolerance sensitivity matrix [S] from

\[
[S]_{3×8} = -[B]_{3×3}^{-1}[A]_{3×8}
\]

\[
[S] = \begin{bmatrix}
\frac{\partial \phi}{\partial x} & \frac{\partial g}{\partial x} & \frac{\partial i}{\partial x} \\
\frac{\partial \phi}{\partial y} & \frac{\partial g}{\partial y} & \frac{\partial i}{\partial y} \\
\frac{\partial \phi}{\partial \theta} & \frac{\partial g}{\partial \theta} & \frac{\partial i}{\partial \theta}
\end{bmatrix}
\]

The values for matrices [A], [B] and [S] are included in the Appendix. The resulting variations for the dependent variables are shown in Table 7.2 and correspond to the nominal component dimensions and tolerances of Table 7.1.

Using the vector approach of the variation polygon, the tolerance sensitivity matrix of the bicycle crank assembly can be directly derived. The variations of the independent variables and the dependent variables are added vectorially in the variation plane to construct the variation polygon as in Figure 7.3. The variations of the independent variables make the loop unclosed if there are no kinematic adjustments. To meet the
assembly constraint, closure must be satisfied. As there are so many variables involved, the collinear vectors $\Delta d$, $\Delta e$ and $\Delta f$ are combined in the polygon forming resultant vector $\Sigma$. The dotted line is the closure error due to the independent variations. The dashed lines represent the kinematic adjustments required to bring the loop closed again. The variations of the angles $\theta$ or $\phi$ produce rotation results that are represented as the product of an effective length multiplied with $\Delta \theta$ or $\Delta \phi$, and the resultant behaves like a vector to contribute to kinematic adjustment.

![Figure 7.3 Variation polygon for bike crank](image)

The tolerance sensitivity can be derived column-by-column from the constructed variation polygons.

<table>
<thead>
<tr>
<th>Column &amp; Column</th>
<th>Variation Due To</th>
<th>Column &amp; Column</th>
<th>Variation Due To</th>
<th>Column &amp; Column</th>
<th>Variation Due To</th>
</tr>
</thead>
<tbody>
<tr>
<td>col. 1 &amp; col. 2</td>
<td>$\Delta a$ (due to $\Delta a$)</td>
<td>col. 3</td>
<td>$\Delta i$ (due to $\Delta c$)</td>
<td>col. 4 &amp; col. 6</td>
<td>$\Delta g$ (due to $\Delta d$)</td>
</tr>
<tr>
<td>a &amp; b</td>
<td>$\Delta b$ or $\Delta a$</td>
<td>c</td>
<td>$\Delta d$</td>
<td>d &amp; f</td>
<td>$\Delta i$ (due to $\Delta d$)</td>
</tr>
<tr>
<td></td>
<td>$\Delta i$ (due to $\Delta a$)</td>
<td></td>
<td>$\Delta c$</td>
<td></td>
<td>$\Delta g$ (due to $\Delta d$)</td>
</tr>
<tr>
<td>col. 7</td>
<td>$\Delta g$ (due to $\Delta h$)</td>
<td>e</td>
<td>$\Delta e$</td>
<td>col. 5</td>
<td>$\Delta g$ (due to $\Delta e$)</td>
</tr>
<tr>
<td>h</td>
<td>$\Delta i$ (due to $\Delta h$)</td>
<td></td>
<td>$\Delta h$</td>
<td></td>
<td>$\Delta g$ (due to $\Delta e$)</td>
</tr>
</tbody>
</table>

![Figure 7.4 Variation polygon for 1st to 7th columns of tolerance sensitivity matrix [S]](image)

From the vector model,

$$\phi + \theta = \pi$$  \hspace{1cm} (7.2)

Differentiating makes
\[ d\phi = -d\theta, \quad \frac{\partial \phi}{\partial \theta} = -1, \quad \text{and} \quad \frac{\partial \phi}{\partial x_i} = 0, \quad x_i \neq \theta \]  

(7.3)

No other variables make the angle \( \phi \) vary except angle \( \theta \). This condition quantifies the \( \phi \) row in the tolerance sensitivity matrix \([S]\).

Derived Tolerance Sensitivity

Variation Polygons due to Independent Variations in Lengths \( a, b, c, d, e, f \) and \( h \)

Columns 1, 2 & 7 (columns \( a, b \) & \( h \))

The dimensions \( a \) and \( b \) have the same orientation, and the tolerance sensitivity for these two variables can be discussed together. From the polygon for \( a \) and \( b \) in Figure 7.4, the increment of dimension \( a \) in a positive direction makes dimension \( i \) decrease and \( g \) increase with the magnitude as shown in the polygon. Direction can be represented by the sign in the tolerance sensitivities and magnitude by the absolute value. The angle \( \phi \) does not vary under the influence of the variation of \( a \). The variation polygon for \( h \) has the same configuration as that for \( a \) and \( b \), except for the flow direction of the vector. The tolerance sensitivity derived from the variation polygon will always conserve the relationship of the sign and the direction of variations. Both variation polygons have the same relationship.

\[
\begin{bmatrix}
\Delta u_i \\
\Delta a
\end{bmatrix} = \begin{bmatrix}
\Delta u_i \\
\Delta b
\end{bmatrix} = -\begin{bmatrix}
\Delta u_i \\
\Delta h
\end{bmatrix}
\]  

(7.4)

Columns 4, 5 & 6 (columns \( d, e \) & \( f \))

Given \( \Delta e \), the variation polygon for \( e \) has the same shape as for \( d \) and \( f \), but is opposite in direction. The tolerance sensitivity is opposite from the variables \( d \) and \( f \).

\[
\begin{bmatrix}
\Delta u_i \\
\Delta d
\end{bmatrix} = \begin{bmatrix}
\Delta u_i \\
\Delta f
\end{bmatrix} = -\begin{bmatrix}
\Delta u_i \\
\Delta e
\end{bmatrix}
\]  

(7.5)

Column 3 (column \( c \))

As the variation of \( c \) does not introduce any variation in the vertical direction, the only kinematic adjustment needed is at the horizontal level.
Variation Polygon due to Variation in Angle $\theta$

Column 8 (column $\theta$)

The effect of the variation of the angle $\theta$ is transformed as the vector by multiplying an effective length. As there is only one independent angle $\theta$, the relationship between $\theta$ and dependent angle $\phi$ is fixed as shown in Equation 7.3. There is a rigid body effect. The length connecting the joint of these two angles is defined as the effective length for counting the interrelationship. Here, "leff" represents the length from dependent joint D (for angle $\phi$) to independent joint H (for angle $\theta$) with the angle $\beta$ relative to the horizontal axis. The closure error due to the variation of the angle $\theta$ produces a vector perpendicular to leff, which has to be closed by the kinematic adjustment $\Delta g$ and $\Delta i$. The variation polygon for this angular variation is shown in Figure 7.5.

\[
\begin{align*}
\text{col. 8} & \quad \theta \\
\frac{\pi}{2} - \theta & \quad \Delta g \text{ (due to } \Delta \theta) \\
\Delta i \text{ (due to } \Delta \theta) & \quad \beta \quad \text{leff} \Delta \theta
\end{align*}
\]

\[
\text{leff} = \sqrt{(e-f-d)^2 + g^2} \quad \beta = \sin^{-1} \frac{a+b-h}{\text{leff}} \\
g = \frac{a+b-h-(e-f-d)\cos(\frac{\pi}{2} - \theta)}{\sin(\frac{\pi}{2} - \theta)}
\]

Figure 7.5 Variation polygon for $\theta$ column of tolerance sensitivity matrix [S]

The geometry in the variation polygon gives us all the information involving the variation of the independent angle $\theta$.

Using effective length and angle $\beta$, the tolerance sensitivity relative to $\theta$ is

\[
\begin{pmatrix}
\Delta \phi \\
\Delta \theta \\
\Delta g \\
\Delta \theta \\
\Delta i \\
\Delta \theta
\end{pmatrix} =
\begin{pmatrix}
-1 \\
\frac{\text{leff} \cos \beta}{\sin(\frac{\pi}{2} - \theta)} \\
\frac{\text{leff} \cos [\beta-\frac{\pi}{2} - \theta]}{\sin(\frac{\pi}{2} - \theta)}
\end{pmatrix}
\begin{pmatrix}
\Delta \phi \\
\Delta \theta \\
\Delta g \\
\Delta \theta \\
\Delta i \\
\Delta \theta
\end{pmatrix} =
\begin{pmatrix}
-1 \\
\frac{\text{DI}}{\cos \theta} \\
-\frac{\text{GH}}{\cos \theta}
\end{pmatrix}
\]

(7.6)
The increase of $\theta$ makes $g$ increase and $i$ decrease. This relationship can be seen very clearly from the directions in variation polygon with the correct magnitude and relationship. For the case where there is only one independent angle and one dependent angle, effective length becomes an important parameter to reveal the interrelationship between the independent and dependent angles. The expression will be simplified further by expressing $\text{leff}$ and $\beta$ in terms of the dimensions in the vector model.

Tolerance Sensitivity Matrix

Based on the above discussion, the tolerance sensitivity can be derived analytically and exactly from all the variation polygons without any data computation involved.

\[
[S] = \begin{bmatrix}
\Delta\phi & \Delta\phi & \Delta\phi & \Delta\phi & \Delta\phi & \Delta\phi & \Delta\phi \\
\Delta a & \Delta b & \Delta c & \Delta d & \Delta e & \Delta f & \Delta h & \Delta\theta \\
\Delta g & \Delta g & \Delta g & \Delta g & \Delta g & \Delta g & \Delta g & \Delta g \\
\Delta a & \Delta b & \Delta c & \Delta d & \Delta e & \Delta f & \Delta h & \Delta\theta \\
\Delta i & \Delta i & \Delta i & \Delta i & \Delta i & \Delta i & \Delta i & \Delta i \\
\Delta a & \Delta b & \Delta c & \Delta d & \Delta e & \Delta f & \Delta h & \Delta\theta
\end{bmatrix}
\]

(7.7)

\[
[S] = [[S]_1][S]_2
\]

(7.8)

\[
[S]_1 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\csc(\frac{\pi}{2} - \theta) & \csc(\frac{\pi}{2} - \theta) & 0 & \cot(\frac{\pi}{2} - \theta) & -\cot(\frac{\pi}{2} - \theta) & \cot(\frac{\pi}{2} - \theta) & -\csc(\frac{\pi}{2} - \theta) & -\cot(\frac{\pi}{2} - \theta) & -\cot(\frac{\pi}{2} - \theta) & -\csc(\frac{\pi}{2} - \theta) & \cot(\frac{\pi}{2} - \theta)
\end{bmatrix}
\]

(7.9)

Submatrix $[S]_2$ has the same expression as shown in Equation 7.6 by the effective length $\text{leff}$ and the angle $\beta$. If using the variables in the vector model, it has the expression as

\[
[S]_2 = \begin{bmatrix}
-1 \\
\frac{(d - e + f)\cdot\sin(\frac{\pi}{2} - \theta) + g\cdot\cos(\frac{\pi}{2} - \theta)}{\sin(\frac{\pi}{2} - \theta)} \\
-\frac{g}{\sin(\frac{\pi}{2} - \theta)}
\end{bmatrix}
\]

(7.10)
The total twenty-four terms in the tolerance sensitivity are handled in six variation polygons. The relationship between the independent and dependent variables is the core for the tolerance sensitivity, and the variation polygon is a very effective tool to handle it. The tolerance sensitivities for the eleven-variable case become very clear through the variation geometry approach.

Design Insights

Important phenomena can be seen from the tolerance sensitivity matrix in Equation 7.9 and 7.10. For the least tolerance sensitivity of i, the angle 0 should be chosen as the smallest value in the allowable range. The smaller 0 is, the smaller the tolerance sensitivity will be. This can be seen from the third row in the tolerance sensitivity matrix. The same conclusion can be drawn for less tolerance sensitivity of g, i.e., the second row of matrix [S]. No other variations can influence angle 0, except 0, which can be seen from the first row. The tolerance sensitivity from the first to the seventh columns will only be influenced by the nominal angle 0, and by no other dimensions.

This information can also be seen geometrically. Figure 7.4 shows that only angle 0 appears in the variation polygon representing the tolerance sensitivity with respect to independent variables a, b, c, d, e, f and h. Figure 7.6 shows that the variation polygon can geometrically monitor the tolerance information. If the nominal angle 0 is chosen as a smaller value, the variations introduced by the same amount of variation of the a or b will be much smaller compared with the case of a larger 0 angle. The comparison can be seen from the geometry.

\[
\begin{align*}
\Delta g \text{ (due to } \Delta a) &= = = = = = \Delta a \\
\Delta i \text{ (due to } \Delta a) &= = = = = = \frac{\pi}{2} - 0 \text{ smaller}
\end{align*}
\]

Figure 7.6 Change of tolerance sensitivity and dependent variation due the nominal change

It is interesting to see that the tolerance sensitivities relative to independent variables a and b are exactly the same, regardless of the difference in the nominal dimensions of a and b. The sensitivity relative to h has the same magnitude but opposite sign, compared with a and b. This is the case for d, f and e. The tolerance sensitivities will be the same for all independent variables having the same orientation relative to the dependent variable.
They should be grouped by the orientation instead of by individual variable. In this way, the effect of independent variations on dependent variations is very simplified and clearly identified.

Since the tolerance sensitivities have been derived graphically and symbolically, we can see more clearly the effect of nominal changes on the tolerance sensitivities. This will give the designer the insights of the tolerance sensitivity from the geometric approach. This knowledge can be obtained from the variation polygon. It can not be obtained from the numerical approach as shown in Equation 7.1.

7.2 EXAMPLE 2: QUICK RETURN MECHANISM

Problem Description

The quick return mechanism shown in Figure 7.7 is used to produce reciprocating movement. The mechanism is composed of a crank, link, slide and ground. The input crank rotates counter-clockwise. At two extreme positions, D and C, the mechanism arrives at the end of the forward stroke and the return stroke. The crank rotates more than 180° for the forward stroke and less than 180° for the return stroke, giving the mechanism its quick return character. The critical assembly feature is the stroke S of the slider, which has application, for example, to piston displacement in a compressor engine. For the analysis of the variation of the displacement, i.e., stroke \( s = u_1 - u_2 \), the mechanism is fixed in the dead-end position to make the problem become a structure. Both positions must be analyzed, since the stroke is the difference.

The model is composed of two closed loops and one open loop. The network graph for closed loops is shown in the following figure. The two closed loops are uncoupled. It is equivalent to two single-loop problems. However, the same variables which appear in both loops are equivalent. The open loop couples the two closed loops to determine their effect on the stroke S.

All the independent and dependent variables are listed in Table 7.3 and 7.4. Associated dependent variations are determined by the linearized method.
Figure 7.7 Vector loops and network graph of quick return mechanism

Table 7.3 Manufactured dimensions for quick return mechanism

<table>
<thead>
<tr>
<th>Variable Name</th>
<th>Description</th>
<th>Basic Size</th>
<th>Tolerance (±)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>Crank length</td>
<td>0.1831 in</td>
<td>0.005 in</td>
</tr>
<tr>
<td>b</td>
<td>Link length</td>
<td>1.0669 in</td>
<td>0.020 in</td>
</tr>
<tr>
<td>h</td>
<td>Height of crank center</td>
<td>0.625 in</td>
<td>0.010 in</td>
</tr>
</tbody>
</table>

Table 7.4 Dependent dimensions for quick return mechanism

<table>
<thead>
<tr>
<th>Variable Name</th>
<th>Description</th>
<th>Basic Size</th>
<th>Variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>φ1</td>
<td>Dependent angle</td>
<td>120.0°</td>
<td>0.76011441°</td>
</tr>
<tr>
<td>φ2</td>
<td>Dependent angle</td>
<td>150.0°</td>
<td>0.76011441°</td>
</tr>
<tr>
<td>u1</td>
<td>Min. x of slide</td>
<td>1.08253175 in</td>
<td>0.0244949</td>
</tr>
<tr>
<td>φ3</td>
<td>Dependent angle</td>
<td>44.9945881°</td>
<td>1.62097965°</td>
</tr>
<tr>
<td>φ4</td>
<td>Dependent angle</td>
<td>134.994588°</td>
<td>1.62097965°</td>
</tr>
<tr>
<td>u2</td>
<td>Max. x of slide</td>
<td>0.62488194 in</td>
<td>0.03082529</td>
</tr>
<tr>
<td>s</td>
<td>Stroke</td>
<td>0.45764981 in</td>
<td>0.01448606</td>
</tr>
</tbody>
</table>
Variation Geometry and Derivation of Tolerance Sensitivity

The variation polygon method is used to analyze the tolerance sensitivity in closed loops.

\[
[S] = \begin{bmatrix}
\frac{\partial \phi_1}{\partial a} & \frac{\partial \phi_1}{\partial b} \\
\frac{\partial \phi_2}{\partial a} & \frac{\partial \phi_2}{\partial b} \\
\frac{\partial \phi_3}{\partial a} & \frac{\partial \phi_3}{\partial b} \\
\frac{\partial \phi_4}{\partial a} & \frac{\partial \phi_4}{\partial b} \\
\frac{\partial u_1}{\partial a} & \frac{\partial u_1}{\partial b} \\
\frac{\partial u_2}{\partial a} & \frac{\partial u_2}{\partial b}
\end{bmatrix}
\]  \hspace{1cm} (7.11)

The relative angles in the vector loops are held constant, except \(\phi_1, \phi_2, \phi_3\) and \(\phi_4\).

By inspection of the triangles ABD and ABC, prescan shows

\[
\phi_1 = \frac{3}{2} \pi - \phi_2 \hspace{1cm} \phi_3 = \phi_4 - \frac{\pi}{2}
\]  \hspace{1cm} (7.12)

Therefore:

\[
d\phi_1 = -d\phi_2 \hspace{1cm} d\phi_3 = d\phi_4
\]  \hspace{1cm} (7.13)

This means that the row representing \(\frac{\Delta \phi_1}{\Delta x_j}\) has the opposite sign to row \(\frac{\Delta \phi_2}{\Delta x_j}\)

\[
\begin{bmatrix}
\Delta \phi_1 \\
\Delta \phi_3
\end{bmatrix} = - \begin{bmatrix}
\Delta \phi_2 \\
\Delta \phi_4
\end{bmatrix} \hspace{1cm} (j = 1, 2, 3)
\]  \hspace{1cm} (7.14)

\[
\begin{bmatrix}
\Delta \phi_3 \\
\Delta \phi_4
\end{bmatrix} = \begin{bmatrix}
\Delta \phi_2 \\
\Delta \phi_4
\end{bmatrix} \hspace{1cm} (j = 1, 2, 3)
\]  \hspace{1cm} (7.15)
Loop 1

Referring to the loop 1, there are three dependent variables: $\phi_1$, $\phi_2$ and $u_1$. The effective length $l_{eff1}$ is BC, i.e., the sum of the link $a$ and $b$. It is the distance between two joints of dependent angles $\phi_1$ and $\phi_2$. The variation polygon for the loop 1 is as follows.

![Variation polygon for the loop 1](image)

Figure 7.8 Variation polygon for the loop 1

For the loop 1, the variation polygons for the tolerance sensitivity are formed by varying one independent dimension at a time, that is, either $\Delta h$, or $\Delta a$, or $\Delta b$, and by determining the corresponding change in $\Delta \phi_2$ and $\Delta u_1$ for each case.

![Variation polygon used for the tolerance sensitivity of loop 1](image)

Figure 7.9 Variation polygon used for the tolerance sensitivity of loop 1

The tolerance sensitivities can be derived from the variation polygons. Each column is derived from one of the variation polygons. The first polygon shows that the $u_1$ and $\phi_2$ decrease if $h$ increases, so negative signs are attached to the corresponding three terms. The tolerance sensitivity matrix from the variation polygons are

\[
[S_1] = \begin{bmatrix}
\Delta \phi_1 & \Delta \phi_1 & \Delta \phi_1 \\
\Delta h & \Delta a & \Delta b \\
\Delta \phi_2 & \Delta \phi_2 & \Delta \phi_2 \\
\Delta h & \Delta a & \Delta b \\
\Delta u_1 & \Delta u_1 & \Delta u_1 \\
\Delta h & \Delta a & \Delta b \\
\end{bmatrix} = \begin{bmatrix}
\frac{\sec(\pi-\phi_2)}{a+b} & \frac{\tan(\phi_2-\frac{\pi}{2})}{a+b} & \frac{\tan(\phi_2-\frac{\pi}{2})}{a+b} \\
\frac{\sec(\pi-\phi_2)}{a+b} & \frac{\cot(\phi_2-\frac{\pi}{2})}{a+b} & \frac{\cot(\phi_2-\frac{\pi}{2})}{a+b} \\
-\frac{\tan(\pi-\phi_2)}{csc(\phi_2-\frac{\pi}{2})} & \frac{\csc(\phi_2-\frac{\pi}{2})}{a+b} & \frac{\csc(\phi_2-\frac{\pi}{2})}{a+b} \\
\end{bmatrix}
\]

(7.16)
Here, since in the extended position, $\Delta a$ and $\Delta b$ act in the same direction.

\[
\begin{bmatrix}
\Delta u_i \\
\Delta a
\end{bmatrix} = \begin{bmatrix}
\Delta u_i \\
\Delta b
\end{bmatrix} \quad (i = 1, 2, 3)
\]  

(7.17)

Loop 2

For the loop 2, the variation polygon for the tolerance sensitivity is

![Variation polygon for loop 2](image)

**Figure 7.10 Variation polygon for loop 2**

BD is effective length $\text{Leff2}$. For each individual independent variation, the variation polygon can be constructed as

![Variation polygons used for tolerance sensitivity of loop 2](image)

**Figure 7.11 Variation polygons used for tolerance sensitivity of loop 2**

The tolerance sensitivities for the loop 2 are derived column-by-column from each variation polygon in Figure 7.11.

\[
[S_2] = \begin{bmatrix}
\Delta \phi_3 & \Delta \phi_3 & \Delta \phi_3 \\
\Delta h & \Delta a & \Delta b \\
\Delta \phi_4 & \Delta \phi_4 & \Delta \phi_4 \\
\Delta h & \Delta a & \Delta b \\
\Delta u_2 & \Delta u_2 & \Delta u_2 \\
\Delta h & \Delta a & \Delta b
\end{bmatrix} = \begin{bmatrix}
\frac{\sec(\pi-\phi_4)}{b-a} & \frac{\ctan(\phi_4 - \frac{\pi}{2})}{b-a} & \frac{\ctan(\phi_4 - \frac{\pi}{2})}{b-a} \\
\frac{\sec(\pi-\phi_4)}{b-a} & \frac{\ctan(\phi_4 - \frac{\pi}{2})}{b-a} & \frac{\ctan(\phi_4 - \frac{\pi}{2})}{b-a} \\
\frac{\sec(\pi-\phi_4)}{b-a} & \frac{\ctan(\phi_4 - \frac{\pi}{2})}{b-a} & \frac{\ctan(\phi_4 - \frac{\pi}{2})}{b-a} \\
\frac{\tan(\pi-\phi_4)}{\csc(\phi_4 - \frac{\pi}{2})} & \frac{\csc(\phi_4 - \frac{\pi}{2})}{\csc(\phi_4 - \frac{\pi}{2})}
\end{bmatrix}
\]  

(7.18)
The second polygon in Figure 7.11 shows that dependent variations decrease when a increases. Looking at the second and third polygons, the directions of a and b are opposite. The two polygons have the same shape, but reversed directions. All the tolerance sensitivities in the a and b columns have opposite signs.

Matrices 7.16 and 7.18 can be combined to construct a complete tolerance sensitivity matrix. The tolerance sensitivity matrix, containing eighteen terms, has been derived from the variation polygon method.

\[
[S] = \begin{bmatrix} [S_1] \\ [S_2] \end{bmatrix}
\]  \hspace{1cm} (7.19)

7.3 EXAMPLE 3: TAPEHUB

The tapehub has one closed loop composed of nine vectors, with eight independent variables and three dependent variables, as shown in Figure 4.1. Figure 7.12 shows the closed vector loop and the network graph for the tapehub.

![Diagram of tapehub](image)

Figure 7.12 Closed vector loop and network graph for tape hub

Angles φ and angle θ have a fixed geometric relationship as

\[
φ = \frac{\pi}{2} - θ
\]  \hspace{1cm} (7.20)

Differentiating,

\[
dφ = -dθ
\]  \hspace{1cm} (7.21)
As \( \phi \) is the dependent angle and \( \theta \) is the independent angle, the \( \phi \) does not change with any other independent variations, except for the change of the angle \( \theta \). For \( \phi \) row in the tolerance sensitivity matrix, '1' is assigned for the tolerance sensitivity of the angle \( \phi \) to angle \( \theta \), and '0' is assigned for others.

**Vector Variation Geometry**

All the variation polygons are shown in the following figure.

![Variation Polygon Diagram]

Figure 7.13 Variation polygon for 1st to 7th columns

**Derived Tolerance Sensitivity**

Variation Polygons due to Variations in Lengths

Columns 1, 6 & 7 (columns b, g & h)

The variation of the dimension b introduces the closure error in a vertical direction, and the adjustments in u and rl are needed to close the loop. The variation polygons due to \( \Delta g \) and \( \Delta h \) have the same shape as those due to \( \Delta b \), except for the reversed directions. This makes the opposite sign in the derived tolerance sensitivity.

Columns 2, 4 & 5 (columns a, e & i)

The independent variables a, e and i are in the horizontal direction in the vector model. Any variation of them only requires a horizontal adjustment, and a change of rl produces a sufficient kinematic adjustment.

\[
\begin{bmatrix}
\Delta u_i \\
\Delta a
\end{bmatrix} = \begin{bmatrix}
\Delta u_i \\
\Delta e
\end{bmatrix} = \begin{bmatrix}
\Delta u_i \\
\Delta i
\end{bmatrix}
\]  

(7.22)

Column 3 (column r)
Since the directions of the variation of the dependent variables in the $\Delta r$ polygon are consistent with vectors defined in the vector model, all the tolerance sensitivities in this column are positive. This means that if $r$ increases as the independent variation, $u$ and $r_l$ will increase as the kinematic adjustments.

Variation Polygon due to Variation in $\theta$

Column 8 (column $\theta$)

- Look at the geometry from vector $u$ to $r$ in the following figure.

$$\text{leff} \cdot \Delta \theta = \sqrt{u^2 + r^2}$$

Figure 7.14 Geometry around angle $\theta$

The variation of the angle $\theta$ introduces rotational variation to all vectors following the current C joint in the loop. The angle $\phi$ is the only dependent angle. The effective length $\text{leff}$ is the line connecting joint C (joint for independent angle $\theta$) and E (joint for dependent angle $\phi$). The effect of the variation of angle $\theta$ is transformed to the vector $\text{leff} \cdot \Delta \theta$ in the variation polygon. The expression $\text{leff} \cdot \Delta \theta$ represents the closure error due to the variation of the angle $\theta$, which needs to be compensated for by the kinematic adjustments. The variation polygon is:

![Variation polygon for the $\theta$ column of tolerance sensitivity matrix](image)

Figure 7.15 Variation polygon for the $\theta$ column of tolerance sensitivity matrix

Here,

$$\alpha = \frac{\pi}{2} - \theta + \tan \left( \frac{r}{u} \right)$$

(7.23)
The increment of the angle $\theta$ due to the manufacturing variation decreases $u$, angle $\phi$ and $rl$. It can be seen from the variation polygon in Figure 7.15, and the tolerance sensitivity can be derived from the variation polygon.

The tolerance sensitivity due to the variation of $\theta$ can be derived from Figure 7.15 as:

$$\begin{pmatrix} \Delta u \\ \Delta \theta \\ \Delta \phi \\ \Delta \theta \\ \Delta rl \\ \Delta \theta \end{pmatrix} = \begin{pmatrix} \frac{2}{\sin \theta} - \frac{\text{leff} \cdot \sin \alpha}{\sin \theta} & -1 & \frac{\text{leff} \cdot \sin \frac{\pi}{2} - \theta + \tan \frac{\pi}{u}}{\sin \theta} \\ -1 & \frac{\text{leff} \cdot \sin \left[\pi - \theta - \alpha\right]}{\sin \theta} & -1 \\ \frac{\text{leff} \cdot \sin \frac{\pi}{2} - \theta + \tan \frac{\pi}{u}}{\sin \theta} & -1 & \frac{\text{leff} \cdot \sin \left[\pi - \theta - \alpha\right]}{\sin \theta} \end{pmatrix}$$

(7.24)

**Tolerance Sensitivity**

The tolerance sensitivity for the tapehub assembly can be derived from all the variation polygons as follows.

$$[S] = \begin{bmatrix} \Delta u & \Delta u & \Delta u & \Delta u & \Delta u & \Delta u & \Delta u \\ \Delta b & \Delta a & \Delta r & \Delta e & \Delta i & \Delta g & \Delta h & \Delta \theta \\ \Delta \phi & \Delta \phi & \Delta \phi & \Delta \phi & \Delta \phi & \Delta \phi & \Delta \phi \\ \Delta b & \Delta a & \Delta r & \Delta e & \Delta i & \Delta g & \Delta h & \Delta \theta \\ \Delta rl & \Delta rl & \Delta rl & \Delta rl & \Delta rl & \Delta rl & \Delta rl & \Delta rl \end{bmatrix}$$

(7.25)

$$[S] = [[S]_1][S]_2$$

(7.26)

$$[S]_1 = \begin{bmatrix} -\sec \left(\frac{\pi}{2} - \theta\right) & 0 & \cot \theta & 0 & 0 & \sec \left(\frac{\pi}{2} - \theta\right) & \sec \left(\frac{\pi}{2} - \theta\right) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\tan \left(\frac{\pi}{2} - \theta\right) & 1 & \csc \theta & 1 & 1 & \tan \left(\frac{\pi}{2} - \theta\right) & \tan \left(\frac{\pi}{2} - \theta\right) \end{bmatrix}$$

(7.27)

$[S]_2$ has expression as shown in Equation 7.22. It can also be expressed in terms of the nominal dimensions as
\[
[S]_2 = \begin{bmatrix}
\sqrt{u^2 + r^2} \cdot \left[\cos\theta \cdot \frac{u}{\sqrt{u^2 + r^2}} + \sin\theta \cdot \frac{r}{\sqrt{u^2 + r^2}}\right] \\
\sin\theta \\
-1 \\
-u \\
\sin\theta
\end{bmatrix}
\]

(7.28)

All the tolerance sensitivities have been completely derived from the variation polygons. The twenty-four terms in the tolerance sensitivity matrix for the closed loop of the tape hub can be handled analytically and exactly by five variation polygons.

7.4 EXAMPLE 4: TRUSS

For cases of more than two dependent angles, additional concepts will be defined. The truss structure will be analyzed using pseudovectors and the frame polygon, which gives a highly compact representation for tolerance sensitivity. This is the first example to demonstrate the development of pseudovector, frame polygon, and their application. The vector loop and network graph for the truss assembly are shown in Figure 7.16.

![Figure 7.16 Vector loop and network graph of truss](image)

**Table 7.5** Manufactured variables for truss

<table>
<thead>
<tr>
<th>Variable Name</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>L1</td>
<td>Length of link</td>
</tr>
<tr>
<td>L2</td>
<td>Length of link</td>
</tr>
<tr>
<td>L3</td>
<td>Length of link</td>
</tr>
</tbody>
</table>

**Table 7.6** Dependent variables for truss

<table>
<thead>
<tr>
<th>Variable Name</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\phi_1)</td>
<td>Dependent angle</td>
</tr>
<tr>
<td>(\phi_2)</td>
<td>Dependent angle</td>
</tr>
<tr>
<td>(\phi_3)</td>
<td>Dependent angle</td>
</tr>
</tbody>
</table>
The manufactured and dependent variables are listed in Table 7.5 and 7.6.

**Pseudovectors and Frame Polygon**

There are three dependent angles which introduce angular variations, if there exist variations in the independent lengths. It was identified in this research that there are other vectors rather than vectors in Figure 7.16, which will connect the tolerance sensitivity with the geometry. They are defined as pseudovectors. In this example, pseudovectors have the same lengths as the vectors in the model, but are opposite in direction. Three corresponding pseudovectors are defined for three dependent angles, as shown in Figure 7.17. The angle \( \phi_1 \) has pseudovector CB. Following the direction of the vector loop in Figure 7.16, the current dependent joint for \( \phi_1 \) is A; the first dependent joint is B; and the second dependent joint C. Pseudovector CB is defined as a vector from the second dependent joint to the first dependent joint after the current dependent joint. It is the side of the triangle opposite to \( \phi_1 \). The angle \( \phi_2 \) has the pseudovector AC, and \( \phi_3 \) has the pseudovector BA. They are summarized in Table 7.7.

<table>
<thead>
<tr>
<th>Dependent Angle</th>
<th>( \phi_1 )</th>
<th>( \phi_2 )</th>
<th>( \phi_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dependent Joint</td>
<td>A</td>
<td>B</td>
<td>C</td>
</tr>
<tr>
<td>Pseudovector</td>
<td>CB</td>
<td>AC</td>
<td>BA</td>
</tr>
</tbody>
</table>

The successive connection of these pseudovectors is defined as the frame polygon for the truss case. The frame polygon is the triangle ACB. The tolerance sensitivity can be directly and easily derived from the pseudovector approach for the three dependent angles.

![Figure 7.17 Frame polygon of the truss](image)

**Projection**

The tolerance sensitivity relative to each independent variable may be expressed as a quotient. The denominator equals the value of twice the area of the frame polygon. The
numerator is the projection of the pseudovector onto the corresponding independent variable, as shown in Figure 7.18 and 7.19.

![Figure 7.18 Projections of pseudovectors CB and AC onto L1, L2 and L3](image1)

![Figure 7.19 Projections of pseudovector BA onto L1, L2 and L3](image2)

For example, the projections of the pseudovector CB of \( \phi_1 \) onto each independent variable are shown in Figure 7.18. The projection onto \( L_1 \) is DB, which has a consistent direction as \( L_1 \), and the positive sign is assigned to the corresponding tolerance sensitivity. The projection onto \( L_2 \) is CB, which is negative with a direction opposite of \( L_2 \). Projection onto \( L_3 \) is CE, which is negative too, as the direction is opposite of \( L_3 \).

It also can be seen that twice the area of the frame polygon gives the absolute value of the determinant of the B matrix, which is the derivative matrix of the loop equations with respect to the dependent variables.

**Tolerance Sensitivity**

The tolerance sensitivity matrix is represented by the following form. Each numerator in the tolerance sensitivity is represented as the projection of pseudovector onto corresponding independent variable. The denominators are represented as twice the area of the frame polygon. For example, \( CBI_L_1 \) represents projection of the pseudovector CB onto \( L_1 \). Note that the pseudovector CB is used to generate the first row of matrix [S], which includes the sensitivities of the \( \phi_1 \) dependent angle. Pseudovectors AC and BA also have a similar relationship to the second and the third rows, i.e., \( \phi_2 \) and \( \phi_3 \) rows.
\[
[S] = \begin{bmatrix}
\Delta \phi_1 & \Delta \phi_1 & \Delta \phi_1 \\
\Delta L_1 & \Delta L_2 & \Delta L_3 \\
\Delta \phi_2 & \Delta \phi_2 & \Delta \phi_2 \\
\Delta L_1 & \Delta L_2 & \Delta L_3 \\
\Delta \phi_3 & \Delta \phi_3 & \Delta \phi_3 \\
\Delta L_1 & \Delta L_2 & \Delta L_3 
\end{bmatrix}
\]  
(7.29)

\[
[S] = \begin{bmatrix}
CB_{L1} & CB_{L2} & CB_{L3} \\
2 ACB & 2 ACB & 2 ACB \\
AC_{L1} & AC_{L2} & AC_{L3} \\
2 ACB & 2 ACB & 2 ACB \\
BA_{L1} & BA_{L2} & BA_{L3} \\
-2 ACB & 2 ACB & 2 ACB 
\end{bmatrix}
\]  
(7.30)

Vector Operation Representation

Equation 7.30 can also be represented by vector operations.

\[
[S] = \begin{bmatrix}
CB \cdot L_1 & CB \cdot L_2 & CB \cdot L_3 \\
[L_1 L_2 \hat{e}] & [L_1 L_2 \hat{e}] & [L_1 L_2 \hat{e}] \\
AC \cdot L_1 & AC \cdot L_2 & AC \cdot L_3 \\
[L_1 L_2 \hat{e}] & [L_1 L_2 \hat{e}] & [L_1 L_2 \hat{e}] \\
BA \cdot L_1 & BA \cdot L_2 & BA \cdot L_3 \\
[L_1 L_2 \hat{e}] & [L_1 L_2 \hat{e}] & [L_1 L_2 \hat{e}] 
\end{bmatrix}
\]  
(7.31)

Here, \([L_1 L_2 \hat{e}]\) is scalar triple product of vectors. \(\hat{e}\) is a unit vector.

\[
[L_1 L_2 \hat{e}] = (L_1 \times L_2) \hat{e}
\]  
(7.32)

\([L_1 L_2 \hat{e}]\) gives the signed value of the volume of the parallelepiped spanned by these three vectors, or the value of the determinant composed of three vectors. The cross product \(L_1 \times L_2\) has the same direction as unit vector \(\hat{e}\). The denominator is the determinant of the geometric sensitivity matrix of dependent variables. Since \(\hat{e}\) is a unit vector, the absolute value of the determinant equals the value of twice the area of the frame polygon.

Additional Properties of the Tolerance Sensitivity Matrix

To obtain the third row of the tolerance sensitivity matrix, the \(\phi_3\) row, two approaches can be used.
1. As a projection of the pseudovector approach

Pseudovector BA is used for the angle $\phi_3$. The tolerance sensitivity can be obtained from the pseudovector of angle $\phi_3$, as was shown in Figure 7.19 and Equation 7.30.

2. As a linear combination of other rows

The tolerance sensitivity of $\phi_3$ can also be obtained from the tolerance sensitivity of $\phi_1$ and $\phi_2$. The dependent angles have a relationship as follows.

$$\phi_1 + \phi_2 + \phi_3 = 2\pi$$  \hspace{1cm} (7.33)

Differentiating,

$$\frac{\partial \phi_1}{\partial \text{any}} + \frac{\partial \phi_2}{\partial \text{any}} + \frac{\partial \phi_3}{\partial \text{any}} = 0$$  \hspace{1cm} (7.34)

$$\frac{\partial \phi_3}{\partial \text{any}} = -\left(\frac{\partial \phi_1}{\partial \text{any}} + \frac{\partial \phi_2}{\partial \text{any}}\right)$$  \hspace{1cm} (7.35)

The tolerance sensitivity $\phi_3$ is a linear combination of $\phi_1$ and $\phi_2$, i.e., the third row is the linear combination of the first and second rows.

This can be seen from the frame polygon also. The pseudovector makes the close relationship of

$$BA + AC + CB = 0$$  \hspace{1cm} (7.36)

$$BA = -(AC + CB)$$  \hspace{1cm} (7.37)

This relationship is shown in the following figure.

![Figure 7.20 Frame polygon and their projection directions for truss](image)

The projection of all three pseudovectors onto any vector direction of independent variables will add to zero. So, the linear combination can be explained from the relationship among the angles or the relationship among the pseudovectors. This case has
three independent and three dependent variables, and the tolerance sensitivity is a square matrix. There is one row tolerance sensitivity, which is the linear combination of the other two rows. Thus, the tolerance sensitivity matrix is a singular matrix for this assembly.

**Relationship to Kinematics Velocity Polygon**

Recalling the analysis of the simplified truss assembly discussed in Chapter 6, Equation 6.6 gives the tolerance sensitivity of the assembly angles with respect to the length L1, as well as the ratio of angular velocities to input linear velocity for the mechanism. The different algebraic forms from the three kinematic inversions of slider-crank mechanism, as were discussed in Chapter 6, can be represented in a single consistent representation by the newly developed concept of pseudovector. This form is shown as the first column of the [S] matrix in Equation 7.30, regardless of the kinematic inversions. This representation is original in nature! The other two columns give representations, if there are any changes in the lengths L2 or L3 for the simplified truss assembly case, or linear velocities \( \frac{dL_2}{dt} \) or \( \frac{dL_3}{dt} \) as inputs for two different slider-crank mechanisms. Each case of slider-crank mechanisms may have various kinematic inversions. However, algebraic representations by pseudovectors are the same for three kinematic inversions. The pseudovector and the projection give a very clear idea about the tolerance sensitivity for this multiple angle assembly. As there is a strong analogy between the velocity and variation analysis, velocity analysis in kinematics of mechanism may benefit from the same geometric insights developed by the variation analysis.

### 7.5 Example 5: Four-Bar Linkage

The four-bar mechanism is a widely used mechanism. The vector loop and network graph are shown in Figure 7.21. In a typical application, the input angle \( \phi_2 \) is coupled to output angle \( \phi_1 \).

In the mechanism case, the derivative of the output angle \( \phi_1 \) relative to each manufacturing dimension reveals the instantaneous position error in the angular displacement due to the inaccuracy of each dimension. In the assembly case, the tolerance sensitivity represents the dependent variation due to the independent variation in the assembly. This common attribute is most interesting for either the mechanism designer or tolerance assembly analyzer.
Frame Polygon and Pseudoeectors

Sensitivities can be derived directly from geometry. Based on the pseudoeectors, frame polygon and projections, all the tolerance

<table>
<thead>
<tr>
<th>Description</th>
<th>Variable Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output angle</td>
<td>( \theta_1 )</td>
</tr>
<tr>
<td>Dependent angle</td>
<td>( \phi_1 )</td>
</tr>
<tr>
<td>Dependent angle</td>
<td>( \phi_4 )</td>
</tr>
<tr>
<td>Dependent angle</td>
<td>( \phi_3 )</td>
</tr>
</tbody>
</table>

Table 7.9 Dependent Variables for Links

<table>
<thead>
<tr>
<th>Description</th>
<th>Variable Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length of output link</td>
<td>( L_4 )</td>
</tr>
<tr>
<td>Length of coupling link</td>
<td>( L_3 )</td>
</tr>
<tr>
<td>Length of input link</td>
<td>( L_2 )</td>
</tr>
<tr>
<td>Length of ground link</td>
<td>( L_1 )</td>
</tr>
<tr>
<td>Input angle</td>
<td>( \phi_0 )</td>
</tr>
</tbody>
</table>

Table 7.8 Manipulability Variables for Links

Figure 7.21 Vector loop of the four-bar

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pseudovector CA (from the second dependent joint C to the first dependent joint A after the current joint D). These dependent angles, joints and pseudovectors are listed in Table 7.10.

![Frame polygon and pseudovectors for four-bar](image)

**Figure 7.22 Frame polygon and pseudovectors for four-bar**

<table>
<thead>
<tr>
<th>Dependent angle</th>
<th>$\phi_1$</th>
<th>$\phi_3$</th>
<th>$\phi_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dependent joint</td>
<td>A</td>
<td>C</td>
<td>D</td>
</tr>
<tr>
<td>Pseudovector</td>
<td>DC</td>
<td>AD</td>
<td>CA</td>
</tr>
</tbody>
</table>

Table 7.10 Dependent angles, joints and pseudovectors for four-bar

The value of twice the area of the frame polygon will be the denominator of the tolerance sensitivity for the angle versus independent variables. The pseudovector is used to define the numerator of the tolerance sensitivity.

**Angle versus Length**

There are four columns of tolerance sensitivities which represent angles $\phi_3$, $\phi_4$ and $\phi_1$ versus lengths L2, L3, L4 and L1.

There are very well behaved geometric patterns for obtaining the variations of the dependent angles to the independent lengths. The numerator for the tolerance sensitivity of the dependent angle is always the projection of the corresponding pseudovector onto the corresponding independent length. The angles between the pseudovectors and the vectors projected decide the signs of the tolerance sensitivities of angles versus the lengths. This means that the directions of the projected vector relative to the direction of the independent dimension vector determine the sign. The denominator of the tolerance sensitivity equals the value of twice the area of the frame polygon. These rules hold for this example, and give a very clear geometric representation for the tolerance sensitivities.
Figure 7.23 Tolerance sensitivity for first column (L2) and first row (ϕ3)

Figure 7.23 shows two examples of the geometric determination of tolerance sensitivities of a dependent angle relative to independent lengths. One explains tolerance sensitivity by column, and the other explains it by row. The first example shows one column of the tolerance sensitivity matrix [S], corresponding to independent variables L2. The projections of pseudovectors for this column are all onto vector L2. The vector to be projected is the corresponding pseudovector for the specific dependent angle as shown in Table 7.10. Pseudovector AD is for ϕ3; CA is for ϕ4; and DC is for ϕ1. The projection of AD onto L2, i.e., FE, has the same direction as the L2 vector, which has been defined in the model. Therefore, the tolerance sensitivity $\frac{\Delta \phi_3}{\Delta L_2}$ is positive. The projection of CA onto L2, i.e., CF, has the opposite direction from that of the L2 vector, and the tolerance sensitivity $\frac{\Delta \phi_4}{\Delta L_2}$ is negative. The projection of DC onto L2, i.e., EC, has the same direction as the L2 vector, and the tolerance sensitivity $\frac{\Delta \phi_1}{\Delta L_2}$ is positive. The procedure is similar for L3, L4 and L1 columns.

The second example in Figure 7.23 shows the row of tolerance sensitivity matrix [S], corresponding to dependent angle ϕ3. The pseudovector for ϕ3, i.e., AD, is projected onto each independent dimension of L2, L3, L4 and L1, and produces FE, GD, AD and AE. They give tolerance sensitivities $\frac{\Delta \phi_3}{\Delta L_2}$, $\frac{\Delta \phi_3}{\Delta L_3}$, $\frac{\Delta \phi_3}{\Delta L_4}$ and $\frac{\Delta \phi_3}{\Delta L_1}$. The procedure is similar for ϕ4 and ϕ1 rows. The pattern from the pseudovector approach is very clear. It can explain the tolerance sensitivity matrix by columns as well as by rows. The magnitude and direction of each vector obtained by the projection of the pseudovector, give the numerator
of the corresponding tolerance sensitivity. The denominator of each term is equal to the value of twice the area of the frame polygon ADC.

**Angle versus Angle**

For sensitivity of a dependent angle versus independent angle, the direction for the projection needs to be introduced. For small changes in \( \phi_2 \), point C moves in a direction perpendicular to L2, we call this the angular projection direction, or the L2 d\( \phi_2 \) direction, as shown in Figure 7.24. The projected vectors, CM for \( \phi_4 \) and KC for \( \phi_1 \), are obtained by projecting their pseudovectors CA and DC onto the L2 d\( \phi_2 \) direction. They correspond to tolerance sensitivities \( \frac{\Delta \phi_4}{\Delta \phi_2} \) and \( \frac{\Delta \phi_1}{\Delta \phi_2} \), as shown in Figure 7.24. The next dependent angle \( \phi_3 \) shares the pseudovector AD with \( \phi_2 \). Again, the denominator of each term equals the value of twice the area of the frame polygon. The derivation is included in the Appendix.

![Figure 7.24 Tolerance Sensitivities from projection for \( \phi_2 \) column](image)

**Projection Space**

The numerator in the derivation is only the projection of the pseudovector onto the corresponding independent vector. The denominator for all terms is derived from the area of the frame polygon. The difference among the five columns in the tolerance sensitivity matrix \([S]\) is only the projections onto different vectors. Pseudovectors are the core for the tolerance sensitivity. They are identified as the controlling factors in the tolerance sensitivity analysis. The independence upon the coordinate of the tolerance sensitivity can be seen from the projection relations. Because the projection space is independent of the coordinate, the tolerance sensitivity is independent of the coordinate.
Figure 7.25 Frame polygon, pseudovectors and their projection space for four-bar

In the analysis, there are more relationships that can be obtained early.

Since, for a four sided quadrilateral,

\[
\phi_2 + \phi_3 + \phi_4 + \phi_1 = 2\pi \tag{7.38}
\]

Differentiating relative to \(\phi_2\)

\[
\frac{\partial \phi_3}{\partial \phi_2} + \frac{\partial \phi_4}{\partial \phi_2} + \frac{\partial \phi_1}{\partial \phi_2} = -1 \tag{7.39}
\]

For the \(\phi_2\) column, the sum of the three tolerance sensitivities equals one.

Likewise, the derivative of Equation 7.38 with respect to any other independent variables gives

\[
\frac{\partial \phi_3}{\partial \phi_2} + \frac{\partial \phi_4}{\partial \phi_2} + \frac{\partial \phi_1}{\partial \phi_2} = 0 \tag{7.40}
\]

For each of the independent length columns, as \(\phi_2\) does not change, any row is the negative summation of the other two rows. This relationship can also be seen from the relationship among the pseudovectors, as we saw for the truss example.

\[AD + CA + DC = 0 \tag{7.41}\]

This shows the linear dependence in the tolerance sensitivity matrix for the case with three dependent angles. In previous truss assembly case, this produces a singular tolerance sensitivity matrix \([S]\). However, for this four-bar case, \([S]\) is not a square matrix. The pseudovectors can be used as the representation of the tolerance sensitivity.
Summary

Two cases appeared in the tolerance sensitivity matrix for multiple dependent angle assemblies. The summary of the pseudovector approach for each case would be:

Angle versus length: pseudovector projection onto independent length over twice the area of the frame polygon.

Angle versus angle: pseudovector projection onto angular projection direction over twice the area of the frame polygon.

Thus, the tolerance sensitivity can be represented in terms of the projection of the pseudovectors and the value of twice the area of the frame polygon. For the four-bar case, it is as follows:

\[
[S] = \begin{bmatrix}
\Delta \phi_3 & \Delta \phi_3 & \Delta \phi_3 & \Delta \phi_3 \\
\Delta \phi_2 & \Delta \phi_1 & \Delta \phi_1 & \Delta \phi_2 \\
\Delta \phi_4 & \Delta \phi_4 & \Delta \phi_4 & \Delta \phi_4 \\
\Delta \phi_2 & \Delta \phi_1 & \Delta \phi_1 & \Delta \phi_2
\end{bmatrix}
\]

\[ (7.42) \]

\[
[S] = \begin{bmatrix}
AD_{L2} & AD_{L3} & AD_{L4} & AD_{L1} \\
2 \text{ADC} & 2 \text{ADC} & 2 \text{ADC} & 2 \text{ADC} \\
\text{CA_{L2}} & \text{CA_{L3}} & \text{CA_{L4}} & \text{CA_{L1}} \\
-2 \text{ADC} & -2 \text{ADC} & -2 \text{ADC} & -2 \text{ADC} \\
\text{DC_{L2}} & \text{DC_{L3}} & \text{DC_{L4}} & \text{DC_{L1}} \\
2 \text{ADC} & 2 \text{ADC} & 2 \text{ADC} & 2 \text{ADC}
\end{bmatrix}
\]

\[ (7.43) \]

In this example, all the tolerance sensitivities for dependent angles have been derived from the pseudovectors. This tolerance sensitivity matrix is not a square matrix. Pseudovector is a very effective tool for variation analysis of multi-angle cases as the geometric patterns are very well behaved. In the next example, we will extend this method to a multiple loop assembly which includes six dependent angles.
7.6 EXAMPLE 6: REMOTE POSITIONING MECHANISM

Figure 7.26 Vector loops and network graph of remote positioning mechanism

The remote Positioning Mechanism has been analyzed in Chapter 4. In that chapter, the tolerance sensitivities were obtained numerically by linearization of the loop equations and matrix algebra. The assembly tolerance sensitivity for closed loops will now be determined by the approach of the vector variation geometry. There are six dependent angles, i.e., $\phi_1, \phi_2, \phi_3, \phi_4, \phi_5$ and $\phi_6$; eight independent lengths, i.e., $a, b, c, d, e, f, g$ and $i$; and four independent angles. Each dependent angle will vary if there is any manufacturing variation. Two closed loops are constructed to solve this multi-loop case, which are ABCDA and CEFGC, as shown above in Figure 7.26.
There are three dependent angles in the loop ABCDA, which are $\phi_1$, $\phi_2$ and $\phi_3$. The variations of the dependent angles due to the variations of the independent lengths $a$, $b$, $c$ and $d$ can be solved within this loop. The loop CEFGC is solved in a similar way. The connection of these two loops is at point C.

**Frame Polygon and Pseudovectors**

There are two frame polygons, which are BDC and EGF. The frame polygon BDC is formed by connecting the three dependent joints for loop ABCDA. The frame polygon EGF is formed by connecting the three dependent joints for loop CEFGC. The pseudovectors for each dependent angle in the loops are defined in Table 7.11.

<table>
<thead>
<tr>
<th>Dependent angle</th>
<th>$\phi_1$</th>
<th>$\phi_2$</th>
<th>$\phi_3$</th>
<th>$\phi_4$</th>
<th>$\phi_5$</th>
<th>$\phi_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dependent joint</td>
<td>B</td>
<td>C</td>
<td>D</td>
<td>E</td>
<td>F</td>
<td>G</td>
</tr>
<tr>
<td>Pseudovector</td>
<td>DC</td>
<td>BD</td>
<td>CB</td>
<td>GF</td>
<td>EG</td>
<td>FE</td>
</tr>
</tbody>
</table>

Both frame polygons are plotted in Figure 7.27, as well as all the pseudovectors.

**Projection**

By the pseudovector approach, the tolerance sensitivities of this multi-loop case can be obtained easily. The denominator of the tolerance sensitivity equals the value of twice the area of the corresponding frame polygon. The numerator is the projection of the pseudovector onto the corresponding independent variable. The sign of the numerator depends upon the direction of the projection and the corresponding independent vector, whether the direction is consistent / positive or otherwise / negative.

**Loops ABCDA and CEFGC**

The relationships among all angles $\phi_1$, $\phi_2$, $\phi_3$ and lengths $a$, $b$, $c$, $d$ can be obtained using the pseudovectors from the loop CEFGC. The relationship between $\phi_4$, $\phi_5$, $\phi_6$ and $e$, $f$, $g$, $i$ are derived from the loop CEFGC.

The tolerance sensitivity for dependent angle $\phi_1$ can be obtained from Figure 7.27.

The pseudovector DC is projected onto dimensions $a$, $b$, $c$ and $d$. The projected values are the numerators of the tolerance sensitivities for first four terms in the
corresponding $\phi_1$ row. The corresponding terms for the $\phi_2$ and $\phi_3$ rows can be found similarly. They are shown in Appendix.

Figure 7.27 Frame polygons, pseudovectors and projection representation of the pseudovector for angle $\phi_1$

Connection

There is interaction between the two loops. It may be seen by inspection that the lengths in loop CEFGC do not affect the angles in loop ABCDA. However, the lengths in ABCDA affect dependent angles in the successive loop. Two loops are weakly coupled. There exists a relationship between the two loops. It can be obtained from the angle
relationship. For angles $\phi_4$, $\phi_5$ and $\phi_6$ and lengths $a$, $b$, $c$ and $d$, i.e., $x_{11}$, which are variables between two loops,

$$\phi_5 - \theta_4 = \phi_2 - \theta_3$$  \hspace{1cm} (7.44)

the relationships among the tolerance sensitivities are

$$\frac{\partial \phi_5}{\partial x_{11}} = \frac{\partial \phi_2}{\partial x_{11}}$$  \hspace{1cm} (7.45)

$$\frac{\partial \phi_4}{\partial x_{11}} = \frac{\partial \phi_6}{\partial x_{11}} = -\frac{\partial \phi_5}{\partial x_{11}}$$  \hspace{1cm} (7.46)

Projection Space

Frame polygons and the projection directions are in Figure 7.28. This figure shows all important elements for the tolerance sensitivity.

![Figure 7.28 Frame polygon and its project directions for remote positioning mechanism](image-url)
Tolerance Sensitivity

The tolerance sensitivities for closed loops can be derived from frame polygon, pseudovectors and the projections as

\[
[S]_{11} = \begin{bmatrix}
\Delta \phi_1 & \Delta \phi_1 & \Delta \phi_1 & \Delta \phi_1 \\
\Delta a & \Delta b & \Delta c & \Delta d \\
\Delta \phi_2 & \Delta \phi_2 & \Delta \phi_2 & \Delta \phi_2 \\
\Delta e & \Delta f & \Delta g & \Delta i \\
\Delta \phi_3 & \Delta \phi_3 & \Delta \phi_3 & \Delta \phi_3 \\
\Delta e & \Delta f & \Delta g & \Delta i \\
\Delta \phi_4 & \Delta \phi_4 & \Delta \phi_4 & \Delta \phi_4 \\
\Delta a & \Delta b & \Delta c & \Delta d
\end{bmatrix}
\begin{bmatrix}
\frac{DC_a}{2 BDC} & \frac{DC_b}{2 BDC} & \frac{DC_c}{2 BDC} & \frac{DC_d}{2 BDC} \\
\frac{BD_a}{2 BDC} & \frac{BD_b}{2 BDC} & \frac{BD_c}{2 BDC} & \frac{BD_d}{2 BDC} \\
\frac{CB_a}{2 BDC} & \frac{CB_b}{2 BDC} & \frac{CB_c}{2 BDC} & \frac{CB_d}{2 BDC}
\end{bmatrix}
\]

(7.47)

\[
[S]_{12} = \begin{bmatrix}
\Delta \phi_1 & \Delta \phi_1 & \Delta \phi_1 & \Delta \phi_1 \\
\Delta e & \Delta f & \Delta g & \Delta i \\
\Delta \phi_2 & \Delta \phi_2 & \Delta \phi_2 & \Delta \phi_2 \\
\Delta e & \Delta f & \Delta g & \Delta i \\
\Delta \phi_3 & \Delta \phi_3 & \Delta \phi_3 & \Delta \phi_3 \\
\Delta e & \Delta f & \Delta g & \Delta i
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

(7.48)

\[
[S]_{21} = \begin{bmatrix}
\Delta \phi_4 & \Delta \phi_4 & \Delta \phi_4 & \Delta \phi_4 \\
\Delta a & \Delta b & \Delta c & \Delta d \\
\Delta \phi_5 & \Delta \phi_5 & \Delta \phi_5 & \Delta \phi_5 \\
\Delta a & \Delta b & \Delta c & \Delta d \\
\Delta \phi_6 & \Delta \phi_6 & \Delta \phi_6 & \Delta \phi_6 \\
\Delta a & \Delta b & \Delta c & \Delta d
\end{bmatrix}
\begin{bmatrix}
\frac{BD_a}{2 BDC} & \frac{BD_b}{2 BDC} & \frac{BD_c}{2 BDC} & \frac{BD_d}{2 BDC} \\
\frac{BD_a}{2 BDC} & \frac{BD_b}{2 BDC} & \frac{BD_c}{2 BDC} & \frac{BD_d}{2 BDC} \\
\frac{BD_a}{2 BDC} & \frac{BD_b}{2 BDC} & \frac{BD_c}{2 BDC} & \frac{BD_d}{2 BDC}
\end{bmatrix}
\]

(7.49)

\[
[S]_{22} = \begin{bmatrix}
\Delta \phi_4 & \Delta \phi_4 & \Delta \phi_4 & \Delta \phi_4 \\
\Delta e & \Delta f & \Delta g & \Delta i \\
\Delta \phi_5 & \Delta \phi_5 & \Delta \phi_5 & \Delta \phi_5 \\
\Delta e & \Delta f & \Delta g & \Delta i \\
\Delta \phi_6 & \Delta \phi_6 & \Delta \phi_6 & \Delta \phi_6 \\
\Delta e & \Delta f & \Delta g & \Delta i
\end{bmatrix}
\begin{bmatrix}
\frac{GF_e}{2 EGF} & \frac{GF_f}{2 EGF} & \frac{GF_g}{2 EGF} & \frac{GF_i}{2 EGF} \\
\frac{EG_e}{2 EGF} & \frac{EG_f}{2 EGF} & \frac{EG_g}{2 EGF} & \frac{EG_i}{2 EGF} \\
\frac{FE_e}{2 EGF} & \frac{FE_f}{2 EGF} & \frac{FE_g}{2 EGF} & \frac{FE_i}{2 EGF}
\end{bmatrix}
\]

(7.50)

\[
[S] = [S]_{11} [S]_{12} [S]_{21} [S]_{22}
\]

(7.51)
Discussion

Some design insights can be seen from the derived tolerance sensitivity matrix. For example, \( \frac{\Delta \phi_4}{\Delta e} \) in the matrix 7.50 can be represented as the ratio of projection of the pseudovector GF of \( \phi_4 \) onto \( e \) over twice the area of the frame polygon EGF, i.e., \( \frac{GF_e}{2 \text{ EGF}} \). If the nominals \( e \) and \( g \) are decreased, area EGF decreases, but \( GF_e \) decreases by the same fraction, so this tolerance sensitivity is independent of the change in \( g \) and \( e \). Similarity exists for \( f \) and \( i \). The tolerance sensitivity matrix previously came by the linearized method in Equation 5.1, from the matrix calculation of the derivative matrices, i.e., geometric sensitivity matrices. It was obtained by numerical method and presented in numerical matrix form. The relationships appear more mysterious in problems with multiple loops or with more independent angles. By the pseudovector and frame polygon, the formulation is beautifully and very well patterned. The important controlling factors are identified and the effects are clearly presented. This can be seen from this remote positioning mechanism, the proceeding cases of truss, four-bar and the following ratchet and pawl examples.

Remote Positioning Mechanism and Watt Mechanism

Topologically, the remote positioning mechanism can be described as a watt mechanism. The Watt mechanism corresponding to the remote positioning mechanism in Figure 4.3 is shown as the first figure in Figure 7.29. The linkage BCE and DCG are ternary linkages. Others are binary. The linkage AD is grounded. This is a Watt I six-bar linkage with one binary input and one binary grounded.

![Figure 7.29 Watt mechanism corresponding to remote positioning mechanism](image)

There are seven joints in the mechanism. Joint A is an independent joint with input angle for mechanism. C is a special joint. It can be seen as a dependent joint for ABCDA loop, and becomes an input joint for CEFGC loop. Other five joints, B, D, E, F and G, are dependent joints. The mechanism can be treated as two loops to be analyzed. The two
corresponding frame polygons are shown in Figure 7.29 also. By proper choice of dimensions, the configuration corresponding to the remote positioning mechanism may be obtained.

7.7 EXAMPLE 7: RATCHET AND PAWL

Problem Description

This assembly is composed of three parts: a ratchet, pawl and housing, or ground. Both ratchet and pawl are attached to the ground. The pawl arm has a cylindrical end that stops the clockwise rotation of the ratchet at increments equal to the spacing of the teeth, and holds it at a specified angular position. The assembly is used to rotationally position an object attached to the ratchet. The assembly, vector model and network graph are shown
above in Figure 7.30. Two closed loops will be required to solve for six dependent variables.

There are seven independent variables in two loops: \( x_p, y_p, d_p, d_r, d_{t1}, d_{t2} \) and \( \theta \). They are six independent lengths and one independent angle. Their names, descriptions, basic sizes and assigned tolerances are listed in Table 7.12. There are six dependent variables: \( \phi_p, \phi_t, \phi_{t2}, \phi_r, d_{s1} \) and \( d_{s2} \). They are four dependent angles and two dependent lengths. Their names, sizes and resultant statistical variations are listed in Table 7.13. All assembly variables are influenced by the manufacturing variations.

<table>
<thead>
<tr>
<th>Variable Name</th>
<th>Description</th>
<th>Basic Size</th>
<th>Tolerance (±)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_p )</td>
<td>X coordinate of pawl (B)</td>
<td>.400 in.</td>
<td>.0015 in.</td>
</tr>
<tr>
<td>( y_p )</td>
<td>Y coordinate of pawl (B)</td>
<td>.300 in.</td>
<td>.0015 in.</td>
</tr>
<tr>
<td>( d_p )</td>
<td>Length of pawl</td>
<td>.300 in.</td>
<td>.001 in.</td>
</tr>
<tr>
<td>( d_{t1}, d_{t2} )</td>
<td>Roller radius of pawl</td>
<td>.050 in.</td>
<td>.001 in.</td>
</tr>
<tr>
<td>( d_r )</td>
<td>Radius of ratchet base</td>
<td>.313 in.</td>
<td>.002 in.</td>
</tr>
<tr>
<td>( \theta )</td>
<td>Angle at root</td>
<td>62.0°</td>
<td>.8°</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Variable Name</th>
<th>Description</th>
<th>Basic Size</th>
<th>Variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d_{s1} )</td>
<td>Distance from face contact</td>
<td>0.08321397 in.</td>
<td>0.00181578</td>
</tr>
<tr>
<td>( d_{s2} )</td>
<td>Distance from back contact</td>
<td>0.08321397 in.</td>
<td>0.00181578</td>
</tr>
<tr>
<td>( \phi_p )</td>
<td>angle of pawl</td>
<td>73.8627189°</td>
<td>0.59114705°</td>
</tr>
<tr>
<td>( \phi_t )</td>
<td>Dependent angle</td>
<td>7.06936713°</td>
<td>0.65025387°</td>
</tr>
<tr>
<td>( \phi_r )</td>
<td>final angle of ratchet</td>
<td>99.0679139°</td>
<td>0.29456926°</td>
</tr>
<tr>
<td>( \phi_{t2} )</td>
<td>Dependent angle</td>
<td>125.06937°</td>
<td>1.19356832°</td>
</tr>
</tbody>
</table>

The interrelationship between the two loops can be seen from the model. Two loops share the common vector \( d_{t1} \) and \( d_{s1} \). The relative angle relationship of the two loops is decided at the point C, which is the branching point. \( \phi_t \) is for the loop ABCDFA, and \( \phi_{t2} \) is for the loop EGCDE. There are three joints, B, C and F, related to the dependent angles.
From Geometric Sensitivity to Tolerance Sensitivity

The loop functions can be obtained for two closed loops. The geometric sensitivity matrices $A$ and $B$ are partial derivatives of loop equations, which represent how much the loop closure condition changes due to the change of the corresponding variables. Here, the subscript $b$ represents the big loop, i.e., ABCDFA, and $s$ represents small loop, i.e., EGCDE.

$$[A] = \begin{bmatrix}
\frac{\partial x_b}{\partial x_b} & \frac{\partial x_b}{\partial y_b} & \frac{\partial x_b}{\partial \phi_b} & \frac{\partial x_b}{\partial \phi_{t2}} & \frac{\partial x_b}{\partial \phi_f} & \frac{\partial x_b}{\partial d\xi_1} & \frac{\partial x_b}{\partial d\xi_2} & \frac{\partial x_b}{\partial d\theta} \\
\frac{\partial x_p}{\partial x_b} & \frac{\partial x_p}{\partial y_b} & \frac{\partial x_p}{\partial \phi_b} & \frac{\partial x_p}{\partial \phi_{t2}} & \frac{\partial x_p}{\partial \phi_f} & \frac{\partial x_p}{\partial d\xi_1} & \frac{\partial x_p}{\partial d\xi_2} & \frac{\partial x_p}{\partial d\theta} \\
\frac{\partial y_b}{\partial x_b} & \frac{\partial y_b}{\partial y_b} & \frac{\partial y_b}{\partial \phi_b} & \frac{\partial y_b}{\partial \phi_{t2}} & \frac{\partial y_b}{\partial \phi_f} & \frac{\partial y_b}{\partial d\xi_1} & \frac{\partial y_b}{\partial d\xi_2} & \frac{\partial y_b}{\partial d\theta} \\
\frac{\partial y_p}{\partial x_b} & \frac{\partial y_p}{\partial y_b} & \frac{\partial y_p}{\partial \phi_b} & \frac{\partial y_p}{\partial \phi_{t2}} & \frac{\partial y_p}{\partial \phi_f} & \frac{\partial y_p}{\partial d\xi_1} & \frac{\partial y_p}{\partial d\xi_2} & \frac{\partial y_p}{\partial d\theta} \\
\frac{\partial \theta_b}{\partial x_b} & \frac{\partial \theta_b}{\partial y_b} & \frac{\partial \theta_b}{\partial \phi_b} & \frac{\partial \theta_b}{\partial \phi_{t2}} & \frac{\partial \theta_b}{\partial \phi_f} & \frac{\partial \theta_b}{\partial d\xi_1} & \frac{\partial \theta_b}{\partial d\xi_2} & \frac{\partial \theta_b}{\partial d\theta} \\
\frac{\partial \theta_p}{\partial x_b} & \frac{\partial \theta_p}{\partial y_b} & \frac{\partial \theta_p}{\partial \phi_b} & \frac{\partial \theta_p}{\partial \phi_{t2}} & \frac{\partial \theta_p}{\partial \phi_f} & \frac{\partial \theta_p}{\partial d\xi_1} & \frac{\partial \theta_p}{\partial d\xi_2} & \frac{\partial \theta_p}{\partial d\theta} \\
\frac{\partial \phi_b}{\partial x_b} & \frac{\partial \phi_b}{\partial y_b} & \frac{\partial \phi_b}{\partial \phi_b} & \frac{\partial \phi_b}{\partial \phi_{t2}} & \frac{\partial \phi_b}{\partial \phi_f} & \frac{\partial \phi_b}{\partial d\xi_1} & \frac{\partial \phi_b}{\partial d\xi_2} & \frac{\partial \phi_b}{\partial d\theta} \\
\frac{\partial \phi_p}{\partial x_b} & \frac{\partial \phi_p}{\partial y_b} & \frac{\partial \phi_p}{\partial \phi_b} & \frac{\partial \phi_p}{\partial \phi_{t2}} & \frac{\partial \phi_p}{\partial \phi_f} & \frac{\partial \phi_p}{\partial d\xi_1} & \frac{\partial \phi_p}{\partial d\xi_2} & \frac{\partial \phi_p}{\partial d\theta} \\
\end{bmatrix} \quad (7.52)$$

$$[B] = \begin{bmatrix}
\frac{\partial x_b}{\partial x_b} & \frac{\partial x_b}{\partial y_b} & \frac{\partial x_b}{\partial \phi_b} & \frac{\partial x_b}{\partial \phi_{t2}} & \frac{\partial x_b}{\partial \phi_f} & \frac{\partial x_b}{\partial d\xi_1} & \frac{\partial x_b}{\partial d\xi_2} & \frac{\partial x_b}{\partial d\theta} \\
\frac{\partial x_p}{\partial x_b} & \frac{\partial x_p}{\partial y_b} & \frac{\partial x_p}{\partial \phi_b} & \frac{\partial x_p}{\partial \phi_{t2}} & \frac{\partial x_p}{\partial \phi_f} & \frac{\partial x_p}{\partial d\xi_1} & \frac{\partial x_p}{\partial d\xi_2} & \frac{\partial x_p}{\partial d\theta} \\
\frac{\partial y_b}{\partial x_b} & \frac{\partial y_b}{\partial y_b} & \frac{\partial y_b}{\partial \phi_b} & \frac{\partial y_b}{\partial \phi_{t2}} & \frac{\partial y_b}{\partial \phi_f} & \frac{\partial y_b}{\partial d\xi_1} & \frac{\partial y_b}{\partial d\xi_2} & \frac{\partial y_b}{\partial d\theta} \\
\frac{\partial y_p}{\partial x_b} & \frac{\partial y_p}{\partial y_b} & \frac{\partial y_p}{\partial \phi_b} & \frac{\partial y_p}{\partial \phi_{t2}} & \frac{\partial y_p}{\partial \phi_f} & \frac{\partial y_p}{\partial d\xi_1} & \frac{\partial y_p}{\partial d\xi_2} & \frac{\partial y_p}{\partial d\theta} \\
\frac{\partial \theta_b}{\partial x_b} & \frac{\partial \theta_b}{\partial y_b} & \frac{\partial \theta_b}{\partial \phi_b} & \frac{\partial \theta_b}{\partial \phi_{t2}} & \frac{\partial \theta_b}{\partial \phi_f} & \frac{\partial \theta_b}{\partial d\xi_1} & \frac{\partial \theta_b}{\partial d\xi_2} & \frac{\partial \theta_b}{\partial d\theta} \\
\frac{\partial \theta_p}{\partial x_b} & \frac{\partial \theta_p}{\partial y_b} & \frac{\partial \theta_p}{\partial \phi_b} & \frac{\partial \theta_p}{\partial \phi_{t2}} & \frac{\partial \theta_p}{\partial \phi_f} & \frac{\partial \theta_p}{\partial d\xi_1} & \frac{\partial \theta_p}{\partial d\xi_2} & \frac{\partial \theta_p}{\partial d\theta} \\
\frac{\partial \phi_b}{\partial x_b} & \frac{\partial \phi_b}{\partial y_b} & \frac{\partial \phi_b}{\partial \phi_b} & \frac{\partial \phi_b}{\partial \phi_{t2}} & \frac{\partial \phi_b}{\partial \phi_f} & \frac{\partial \phi_b}{\partial d\xi_1} & \frac{\partial \phi_b}{\partial d\xi_2} & \frac{\partial \phi_b}{\partial d\theta} \\
\frac{\partial \phi_p}{\partial x_b} & \frac{\partial \phi_p}{\partial y_b} & \frac{\partial \phi_p}{\partial \phi_b} & \frac{\partial \phi_p}{\partial \phi_{t2}} & \frac{\partial \phi_p}{\partial \phi_f} & \frac{\partial \phi_p}{\partial d\xi_1} & \frac{\partial \phi_p}{\partial d\xi_2} & \frac{\partial \phi_p}{\partial d\theta} \\
\end{bmatrix} \quad (7.53)
The tolerance sensitivity matrix can be calculated by the matrix method using Equation 5.1. In the multi-loop case, the numerator and the denominator of each tolerance sensitivity may not be derived from the same loop, due to coupling between loops.

For the ratchet and pawl case, the tolerance sensitivity would represent the relationship as

\[
[S] = \begin{bmatrix}
\frac{\partial \phi_p}{\partial x_p} & \frac{\partial \phi_p}{\partial y_p} & \frac{\partial \phi_s}{\partial x_p} & \frac{\partial \phi_s}{\partial y_p} & \frac{\partial \phi_t}{\partial x_p} & \frac{\partial \phi_t}{\partial y_p} & \frac{\partial \phi_p}{\partial d_t} & \frac{\partial \phi_p}{\partial d_t} & \frac{\partial \phi_p}{\partial \theta} \\
\frac{\partial \phi_t}{\partial x_p} & \frac{\partial \phi_t}{\partial y_p} & \frac{\partial \phi_t}{\partial d_p} & \frac{\partial \phi_t}{\partial d_r} & \frac{\partial \phi_t}{\partial d_t} & \frac{\partial \phi_t}{\partial d_t} & \frac{\partial \phi_t}{\partial d_t} & \frac{\partial \phi_t}{\partial d_t} & \frac{\partial \phi_t}{\partial \theta} \\
\frac{\partial \phi_t}{\partial x_p} & \frac{\partial \phi_t}{\partial y_p} & \frac{\partial \phi_t}{\partial d_p} & \frac{\partial \phi_t}{\partial d_r} & \frac{\partial \phi_t}{\partial d_t} & \frac{\partial \phi_t}{\partial d_t} & \frac{\partial \phi_t}{\partial d_t} & \frac{\partial \phi_t}{\partial d_t} & \frac{\partial \phi_t}{\partial \theta} \\
\frac{\partial \phi_t}{\partial x_p} & \frac{\partial \phi_t}{\partial y_p} & \frac{\partial \phi_t}{\partial d_p} & \frac{\partial \phi_t}{\partial d_r} & \frac{\partial \phi_t}{\partial d_t} & \frac{\partial \phi_t}{\partial d_t} & \frac{\partial \phi_t}{\partial d_t} & \frac{\partial \phi_t}{\partial d_t} & \frac{\partial \phi_t}{\partial \theta} \\
\frac{\partial \phi_t}{\partial x_p} & \frac{\partial \phi_t}{\partial y_p} & \frac{\partial \phi_t}{\partial d_p} & \frac{\partial \phi_t}{\partial d_r} & \frac{\partial \phi_t}{\partial d_t} & \frac{\partial \phi_t}{\partial d_t} & \frac{\partial \phi_t}{\partial d_t} & \frac{\partial \phi_t}{\partial d_t} & \frac{\partial \phi_t}{\partial \theta} \\
\frac{\partial \phi_t}{\partial x_p} & \frac{\partial \phi_t}{\partial y_p} & \frac{\partial \phi_t}{\partial d_p} & \frac{\partial \phi_t}{\partial d_r} & \frac{\partial \phi_t}{\partial d_t} & \frac{\partial \phi_t}{\partial d_t} & \frac{\partial \phi_t}{\partial d_t} & \frac{\partial \phi_t}{\partial d_t} & \frac{\partial \phi_t}{\partial \theta} \\
\frac{\partial \phi_t}{\partial x_p} & \frac{\partial \phi_t}{\partial y_p} & \frac{\partial \phi_t}{\partial d_p} & \frac{\partial \phi_t}{\partial d_r} & \frac{\partial \phi_t}{\partial d_t} & \frac{\partial \phi_t}{\partial d_t} & \frac{\partial \phi_t}{\partial d_t} & \frac{\partial \phi_t}{\partial d_t} & \frac{\partial \phi_t}{\partial \theta} \\
\frac{\partial \phi_t}{\partial x_p} & \frac{\partial \phi_t}{\partial y_p} & \frac{\partial \phi_t}{\partial d_p} & \frac{\partial \phi_t}{\partial d_r} & \frac{\partial \phi_t}{\partial d_t} & \frac{\partial \phi_t}{\partial d_t} & \frac{\partial \phi_t}{\partial d_t} & \frac{\partial \phi_t}{\partial d_t} & \frac{\partial \phi_t}{\partial \theta} \\
\frac{\partial \phi_t}{\partial x_p} & \frac{\partial \phi_t}{\partial y_p} & \frac{\partial \phi_t}{\partial d_p} & \frac{\partial \phi_t}{\partial d_r} & \frac{\partial \phi_t}{\partial d_t} & \frac{\partial \phi_t}{\partial d_t} & \frac{\partial \phi_t}{\partial d_t} & \frac{\partial \phi_t}{\partial d_t} & \frac{\partial \phi_t}{\partial \theta}
\end{bmatrix}
\]

(7.54)

Each term in the matrix of tolerance sensitivity represents the amount of sensitivity of dependent variation due to the independent variation. The relationship between the dependent variation and the independent variation looks very complex in such a two loop problem, as it is the product of the inverse matrix of \([B]\) (which deals with the adjoint matrix and the determinant of the \([B]\) matrix) and the \([A]\) matrix. The pseudovector approach makes the relationship more clear.

**Prescan for Geometric Representation of Tolerance sensitivity**

The tolerance sensitivities in this two loop ratchet and pawl case have forty-two relationships between variation of the independent and the dependent variables. From geometry, some of the terms can be eliminated from the analysis through a prescan procedure.

Because of the tangential geometry between \(dt1\) and \(ds1\), \(dt2\) and \(ds2\), the configuration of quadrilateral EGCDE becomes rigid if there are no variations introduced
from \(d_1\), \(d_2\) and \(\theta\). The variations in \(x_p\), \(y_p\), \(d_p\) and \(d_r\) cannot introduce variations of \(d_1\), \(d_2\) and \(\phi_{12}\). This gives twelve terms as zeros in the tolerance sensitivity matrix \([S]\), corresponding to \(x_p\), \(y_p\), \(d_p\) and \(d_r\) columns, and \(d_1\), \(d_2\) and \(\phi_{12}\) rows. The angle \(\theta\) always equals \(\phi_{12}\) geometrically. They define some zero coupling terms in the tolerance sensitivity matrix. The angle \(\phi_t\), as a parameter outside this quadrilateral, is possible to change due to other variations. Only twenty-seven terms in the tolerance sensitivity need to be further analyzed.

As three of the dependent variables, i.e., \(\phi_{12}\), \(d_1\) and \(d_2\), are not influenced by the independent variables in the loop ABCDFA, the only terms in the tolerance sensitivity, which are influenced, are \(\phi_p\), \(\phi_t\) and \(\phi_f\). There are three dependent joints and they become the key factors in the analysis later.

**Tolerance Sensitivity by Frame Polygon, Pseudovectors and Projections**

The pseudovectors for three dependent angles are defined in the following Table 7.14.

<table>
<thead>
<tr>
<th>Dependent angle</th>
<th>(\phi_p)</th>
<th>(\phi_t)</th>
<th>(\phi_f)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dependent joint</td>
<td>B</td>
<td>C</td>
<td>F</td>
</tr>
<tr>
<td>Pseudovector</td>
<td>FC</td>
<td>BF</td>
<td>CB</td>
</tr>
</tbody>
</table>

The frame polygon BFC is composed of three pseudovectors FC, BF and CB. The tolerance sensitivity of the dependent angles versus independent lengths can be represented by the pseudovector approach. The numerator of the tolerance sensitivity can be represented as the projected length of the pseudovector onto the direction of the independent variables considered. Pseudovectors are determined by the dependent joints. For example, following the direction of vector loop in Figure 7.30, joint C is the joint of the first dependent angle after the current joint B of \(\phi_p\), and F is the joint of second dependent angle. The vector connecting these two joints from the second to the first forms the pseudovector FC for angle \(\phi_p\). It is not a component dimension. It is neither an independent nor dependent variable. However, it becomes very important for the dependent angle \(\phi_p\), as it contains geometric information about tolerance sensitivity of angle versus length. The projection onto corresponding independent variables is used in the numerator of tolerance sensitivity with magnitude and sign. The value of twice the area of
the frame polygon is used as the denominator of the tolerance sensitivity. The following plot shows $\frac{\partial \phi_p}{\partial x_p}$.

Figure 7.31 Geometric correspondence of tolerance sensitivity of type 1

FC has the projection FH in the xp direction, as shown in the dotted line with the direction consistent with the xp vector direction. The corresponding tolerance sensitivity can be expressed as:

$$\frac{\Delta \phi_p}{\Delta x_p} = \frac{FC \times \cos(AFC)}{2 \times BFC} = \frac{FC_{xp}}{2 \times BFC} \quad (7.55)$$

The projection of pseudovector FC of $\phi_p$ onto other dimensions, yp, dp and dr, follow the same patterns, and they can be very easily obtained. Similarly, the application of pseudovectors BF for $\phi_1$ and CB for $\phi_2$ gives a very direct and simplified representation of tolerance sensitivities for corresponding rows. The denominator equals the value of twice the area of the frame polygon.
Independent lengths dt1 and dt2 pass through the joint C, where there are two dependent angles. An antiprojection is needed, as shown in Figure 7.32. Pseudovector FC is first projected onto the direction of ds2, then it is reverse projected onto direction dt1. For angle versus angle case, the angle is multiplied by the corresponding length. The dotted line in the following plot shows the numerator of the tolerance sensitivity $\frac{\partial \phi_p}{\partial dt1}$.

![Diagram showing geometric correspondence of tolerance sensitivity of type 2](image)

**Figure 7.32 Geometric correspondence of tolerance sensitivity of type 2**

$$\frac{\Delta \phi_p}{\Delta dt1} = -\frac{FC \cdot \cos(CFH) / \sin \theta}{2 \cdot BFC} = -\frac{FC_{ds2/dt1}}{2 \cdot BFC} \quad (7.56)$$

For the third row, the angle $\phi_{t2}$ equals $\theta$. The variation of $\phi_{t2}$ to variation of $\theta$ is one, and all other terms in the third row, i.e., $\phi_t$ row, are zeros.

The fourth row, i.e., $\phi_f$ row, has a similar frame polygon representation. The pseudovector is CB for this row. It also can be seen that the fourth row is the liner combination of the first row and the second row. Because of

$$\phi_f + \phi_p + \phi_t = \pi \quad (7.57)$$
Therefore,

\[
\frac{\partial \phi_p}{\partial \text{inde}} = - \left( \frac{\partial \phi_p}{\partial \text{inde}} - \frac{\partial \phi_t}{\partial \text{inde}} \right)
\]  

(7.58)

This linear dependent relationship can also be obtained from the pseudovectors.

\[
\text{FC} + \text{CB} + \text{BF} = 0
\]  

(7.59)

Not all terms are derived from pseudovectors for this example problem. The fifth and sixth rows have variation polygon representations as the relationship of variation of length versus length. Here, ds1 and ds2 are only influenced by dt1, dt2 and \( \theta \).

If looking at the columns, the numerators of tolerance sensitivities in the [S] matrix are parallel to the specified independent length variables in the graphic representation. For example, dependent angle \( \phi_p \) has pseudovector FC, and projected vector is FH. FH is parallel to \( x_p \) and produces the numerator of tolerance sensitivity \( \frac{\Delta \phi_p}{\Delta x_p} \). If FC is projected onto \( y_p \), the result projected vector is HC, which is parallel to \( y_p \). This was shown in Figure 7.31.

\[
[S] = \begin{bmatrix}
\Delta \phi_p & \Delta \phi_p & \Delta \phi_p & \Delta \phi_p & \Delta \phi_p & \Delta \phi_p \\
\Delta x_p & \Delta y_p & \Delta d_p & \Delta d_r & \Delta d t_1 & \Delta d t_2 & \Delta \theta \\
\Delta \phi_t & \Delta \phi_t & \Delta \phi_t & \Delta \phi_t & \Delta \phi_t & \Delta \phi_t \\
\Delta x_p & \Delta y_p & \Delta d_p & \Delta d_r & \Delta d t_1 & \Delta d t_2 & \Delta \theta \\
\Delta \phi_{t2} & \Delta \phi_{t2} & \Delta \phi_{t2} & \Delta \phi_{t2} & \Delta \phi_{t2} & \Delta \phi_{t2} \\
\Delta x_p & \Delta y_p & \Delta d_p & \Delta d_r & \Delta d t_1 & \Delta d t_2 & \Delta \theta \\
\Delta \phi_f & \Delta \phi_f & \Delta \phi_f & \Delta \phi_f & \Delta \phi_f & \Delta \phi_f \\
\Delta x_p & \Delta y_p & \Delta d_p & \Delta d_r & \Delta d t_1 & \Delta d t_2 & \Delta \theta \\
\Delta \phi_{s1} & \Delta \phi_{s1} & \Delta \phi_{s1} & \Delta \phi_{s1} & \Delta \phi_{s1} & \Delta \phi_{s1} \\
\Delta x_p & \Delta y_p & \Delta d_p & \Delta d_r & \Delta d t_1 & \Delta d t_2 & \Delta \theta \\
\Delta \phi_{s2} & \Delta \phi_{s2} & \Delta \phi_{s2} & \Delta \phi_{s2} & \Delta \phi_{s2} & \Delta \phi_{s2} \\
\Delta x_p & \Delta y_p & \Delta d_p & \Delta d_r & \Delta d t_1 & \Delta d t_2 & \Delta \theta \\
\end{bmatrix}
\]  

(7.60)

\[ [S] = [[S]_1 [S]_2] \]  

(7.61)
\[
[S]_1 = \begin{bmatrix}
\frac{FC_{xp}}{2 \text{ BFC}} & \frac{FC_{yp}}{2 \text{ BFC}} & \frac{FC_{dp}}{2 \text{ BFC}} & \frac{FC_{dr}}{2 \text{ BFC}} \\
\frac{BFC_{xp}}{2 \text{ BFC}} & \frac{BFC_{yp}}{2 \text{ BFC}} & \frac{BFC_{dp}}{2 \text{ BFC}} & \frac{BFC_{dr}}{2 \text{ BFC}} \\
2 \text{ BFC} & 2 \text{ BFC} & 2 \text{ BFC} & 2 \text{ BFC} \\
0 & 0 & 0 & 0 \\
\frac{CB_{xp}}{2 \text{ BFC}} & \frac{CB_{yp}}{2 \text{ BFC}} & \frac{CB_{dp}}{2 \text{ BFC}} & \frac{CB_{dr}}{2 \text{ BFC}} \\
2 \text{ BFC} & 2 \text{ BFC} & 2 \text{ BFC} & 2 \text{ BFC} \\
0 & 0 & 0 & 0
\end{bmatrix}
\] (7.62)

\[
[S]_2 = \begin{bmatrix}
\frac{-FC_{d2/dt1}}{2 \text{ BFC}} & \frac{-FC_{d1/dt1}}{2 \text{ BFC}} & \frac{FC_{d1/d32 - \theta}}{2 \text{ BFC}} \\
\frac{-BFC_{d2/dt1}}{2 \text{ BFC}} & \frac{BFC_{d1/dt1}}{2 \text{ BFC}} & \frac{-BFC_{d1/d32 - \theta}}{2 \text{ BFC}} \\
2 \text{ BFC} & 2 \text{ BFC} & 1 \\
0 & 0 & 1 \\
\frac{-CB_{d2/dt1}}{2 \text{ BFC}} & \frac{CB_{d1/dt1}}{2 \text{ BFC}} & \frac{-CB_{d1/d32 - \theta}}{2 \text{ BFC}} \\
2 \text{ BFC} & 2 \text{ BFC} & 2 \text{ BFC} \\
\csc \theta & \cotan \theta & -dt1 - ds1 \cdot \csc \theta \\
\csc \theta & \cotan \theta & -ds1 \cdot \csc \theta
\end{bmatrix}
\] (7.63)

### 7.8 NEW CONCEPT AND TERMINOLOGY

New approaches through vector variation geometry have been developed in Chapters 5 through 7. From this new geometric and analytic approach, the tolerance sensitivities and assembly variations can be defined from geometric features from case studies. The new concept and terminology are summarized and defined as follows.

**Variation polygon:** a polygon which is composed of the vectors representing the variations of independent and dependent variables. It contains the variation relationships. It can be used to derive the tolerance sensitivities for assembly.

**Dependent joint:** joint for dependent angle. It is a basic element in forming the frame polygon to represent tolerance sensitivity of angle versus length. It reflects the angular kinematic adjustment if the independent length changes.

**Pseudovector:** a vector jointing two dependent joints in an assembly loop. While each vector in the vector model or loop equation is always associated with a single component dimension, pseudovector may not be. It is used to find the numerator of the tolerance sensitivity for the variation of the angle to the variation of the length. In loops
containing three dependent angles, it is the vector connecting the second dependent joint to the first dependent joint after the current dependent joint, if the counter takes direction from the established model of the vector loop, as shown in previous examples.

Frame polygon: polygon that only connects the dependent joints of a vector loop. The sides are pseudovectors. The value of twice the area of the polygon equals the denominator of the tolerance sensitivities for the variation of an angle relative to the variation of a length.

Projection: projection of a pseudovector onto a corresponding independent variable. The direction and the magnitude of projection are used to represent the numerator of the tolerance sensitivity. The positive sign is taken if projection has the same direction as the variable vector concerned, and vice versa.

Antiprojection: a combination of a projection of a pseudovector onto an intermediate direction, followed by a reverse projection onto the final desired direction of the independent variable.

7.9 SUMMARY

This chapter presented new ways to obtain tolerance sensitivity and variation for mechanical assemblies. Vector variation geometry includes variation polygon and frame polygon. It is an innovative approach to represent the tolerance sensitivity. This chapter derived algebraic expressions for the tolerance sensitivity matrices and the variations through this approach. The variation polygons are used in loops with less than two dependent angles. They are constructed by variation vectors. Tolerance sensitivities and assembly variations can be derived from them. The frame polygons are used in loops with three dependent angles and constructed by pseudovectors. Each pseudovector is obtained by joining two dependent joints. The projection of pseudovector and area of frame polygon can be used to derive tolerance sensitivities.

The concept of kinematic adjustment due to closure constraints is key for tolerance analysis. A spectrum of case studies were analyzed. The implicit relationships between the manufacturing and assembly variables have been made explicitly through vector variation geometry. They have been represented geometrically and derived analytically. Variation polygon and frame polygon are effective methods in revealing the relationship between the independent and dependent variables and learning the nature of tolerance sensitivity. Recommendations will be in the next chapter.
Chapter 8

CONCLUSIONS AND RECOMMENDATIONS

This research has developed a variety of methods to accomplish the objectives. The methods include both qualitative and quantitative analyses of tolerance sensitivity and variations, which are suitable for both preliminary and detailed design stages. They will provide more complete analysis than has been previously attained. The tolerance sensitivity and assembly variations can be described by relationships, which are particularly useful, as they can accommodate changes in the nominal dimensions. New methods can effectively represent tolerance sensitivity and variations in assemblies. Also, they will make analysis more efficient in terms of computations.- In addition, alternative designs may be compared and evaluated. The results of this research should simplify tolerance analysis and encourage design modifications.

This chapter summarizes the important contributions made by this research and presents a discussion of the significance of the new methods, along with several conclusions. Finally, recommendations are made for future research possibilities in the field of geometric and analytical approaches, as well as some possible applications for tolerance analysis of mechanical assemblies.

8.1 CONTRIBUTIONS

Specific contributions are listed below:

- Three new methods for evaluating manufacturability of new product design were developed. They are based on the representation of a range of possible solutions as a surface in a multi-dimensional design space. By applying assembly constraints and production tolerances throughout the space, the effects of nominal dimensions on the variation of critical assembly features can be assessed. Several case studies are presented.

- The real surface method integrates statistical tolerance analysis with nonlinear optimization techniques. After the designer defines the assembly model, Opt-Tol performs automatic movement between modeling and tolerance analysis, explores the design space, provides detailed surface information, and searches for the optimum. The results are used as standard reference values.
• The VRS method combines aspects of response surface methodology with linear statistical tolerance analysis of mechanical assemblies. Nonlinear and linear computations are minimized by fitting a variation surface through a set of sample solution points and applying optimization to the fitted surface. The reduced computation requirements and approximate nature of this method make it well suited for estimating the manufacturability of a design early in the design process.

• The QV method uses the derivative of the quadratic nominal surface to obtain an estimation of the variation surface. It is the most efficient computationally, but also the most limited.

• The application of surface methods to compare the manufacturability of two alternative designs was demonstrated, as an illustration of the value of these tools for evaluating designs in the early stage.

• The discovery of Vector Variation Geometry, which is a totally new direction for assembly tolerance analysis methods, includes the variation polygon and frame polygon. They provide both quantitative and qualitative information about tolerance sensitivities.

• The development of the variation polygon is an original contribution to assembly tolerance analysis methods. The results are both graphical and analytical, since the polygon is a vector representation of the algebraic expressions of variation.

• The development of the pseudovectors and frame polygon, which are used for angular sensitivities in assemblies with more than two dependent angles, can directly relate the angular variation sensitivities to the assembly geometry. They can be used to provide very compact representations for the tolerance sensitivities.

• Polygon approaches were applied to directly obtain tolerance sensitivities from the nominal geometry of an assembly. The relationship between the independent variables and dependent variables described by implicit kinematic assembly constraint equations becomes explicit. Closed form algebraic expressions can be obtained, rather than the numerically determined tolerance sensitivities of previous methods. No intensive modeling and calculations are needed for design alternatives of nominal dimensions.
• The first graphical representation and verification for tolerance sensitivities will provide visual feedback to facilitate tolerance analysis. This was not previously available. The development will provide greater insight for tolerance analysis.

• The discovery of the parallels or analogy between tolerance analysis and kinematics will help to understand the nature of variation and tolerance sensitivity. The connection between variation and velocity will facilitate both fields in the future. The polygons are tools not only for mechanical assemblies, but for mechanism synthesis and kinematic analysis as well.

• The compilation of a number of typical assembly case studies and their polygons showed the correspondence between tolerance sensitivity and geometric assembly configuration. This will be a valuable contribution to the field of tolerance analysis as well kinematics. It will provide groundwork from which further research can be done.

8.2 CONCLUSIONS

This dissertation has presented new methods for variation analysis. Their application will improve current analysis methods and facilitate evaluation of product manufacturability earlier in the design process with great benefits to industry.

The comparisons of the three methods for evaluating assembly variation over a region of design space are summarized in Tables 4.28 through 4.30. The full evaluation of the statistical variations involves the solution of the nonlinear system of equations for the nominal assembly dimensions, and the linearized solution for the statistical variations. The solution is needed for each experimental point in the VRS method, and for every search point in Opt-Tol. QV only needs the quadratic nominal assembly model for the evaluation of derivatives, but some tolerance sensitivities cannot be obtained. From the amount of computation, QV uses the least, VRS uses a moderate amount, and Opt-Tol uses the most.

Comparing the accuracy of the methods, Opt-Tol is the most accurate. VRS gives close estimation for both variations and design dimensions compared with Opt-Tol. QV may provide some estimation if the number of the variation equals the order of the fitting of the nominal. However, the number of experiment points makes the nominal fitting for a higher order prohibitive. VRS gives good estimation results in terms of the effort and the accuracy.
The VRS and QV methods can estimate design dimensions for variations at the early design stage. These design dimensions may be used to produce a design which is robust to variation or may be used to select among competing designs for the most manufacturable. They are approximate. In later stages, where great accuracy is needed, these design dimensions can be used to evaluate accurate variation values for the designer.

The analysis procedures are usually numerical in previous research. New geometric approaches have been developed in this dissertation. The vector variation geometry includes the variation polygon and frame polygon. The research shows that the geometric approach is significantly different from the numerical approach. It is superior in revealing the nature of variation in mechanical assemblies and the relationship to kinematic adjustments. It helps to identify and develop an understanding of the mechanism governing tolerance sensitivities and variations.

The variation polygon and frame polygon developed in this dissertation have been successfully used in tolerance analysis. They are very effective. They are relationship-oriented rather than data-oriented. The results for tolerance sensitivities and assembly variations can be graphical, analytical, as well as numerical. The direction and magnitude can be obtained from the geometric features. The feedback about the kinematic adjustment processes in a mechanical assembly can be fast, accurate and visual.

The variation polygon gives significant insight on how kinematic adjustments are performed and how variations are produced as consequences of the manufacturing variations. In the polygon approaches, the relationship between the independent and dependent variables is preserved. This is especially important at the early design stage, since the decisions for nominal dimensions are difficult to make due to the lack of tolerance information. Tolerance analysis by the data-oriented approach is prohibitive at the design stage due to the need for repeating the modeling process. However, by the polygon approach, little effort needs to be added to consider design alternatives of different nominal dimensions. The polygon approach permits tolerance analysis even before the nominal dimensions are finalized. It will have potential impact on tolerance analysis in terms of computation time, design iteration, objectivity, efficiency, visibility, and capability for the early design stage as well as later.

It was discovered that there exist some dimensions, neither manufactured nor assembly dimensions, which will influence or control the variations of the dependent angles or angular variations. The dependent joints are very important in controlling
variations of angles. The pseudovectors and frame polygon produce an effective way to analyze the tolerance sensitivities of more than two dependent angles. The case studies show that the prescan is a necessary step to simplify angle analysis.

This dissertation has made a thorough study of the geometric approach to variation analysis and design in assemblies. The case studies demonstrated the effectiveness of this approach. They show that results are consistent with current numerical approaches. The polygon approaches avoid the tedious calculation of derivatives by matrix methods. This is the first analysis and exposure of the geometric features of variation analysis in mechanical assemblies. The principle can be used in the geometric, form, and all additive variation sources. The developments in the geometric approach facilitate not only the mechanical assembly, but kinematics as well. They have far-reaching implications.

8.3 RECOMMENDATIONS

Much of the work done has been to develop methods to facilitate early design methods for tolerance analysis. The research in the dissertation provided the groundwork for tolerance analysis from a geometric and analytical approach. It is promising. However, there are still some areas open for future research.

As the kinematics of mechanism analysis is a well-developed field, the analogy between the kinematics and tolerance analysis should be investigated further. It will help the development of tolerance analysis. More theoretical studies along these lines should be continued for variation analysis. The results will benefit not only assembly tolerance analysis, but also kinematics as well.

Further graphical implementation of the variation polygon should be made. It can include the polygons as well as the worst case, statistical and exact analysis. It is the micro analysis for variations, i.e., in small scale. This will give the designer a direct visual feedback from the tolerance view.

The nominal design and tolerance design, i.e., variational system and tolerance analysis system should be integrated. The macro change in the nominal can be combined with the micro change in the nominal. Dynamic kinematic adjustment in tolerance analysis can be monitored in the design process through the polygons. The consequences can be reported. It can provide information in geometrical, analytical, as well as numerical form. This can facilitate the parallel design process of the nominal design and tolerance design.
Tolerance allocation and the cost analysis can be added for a complete analysis, since all independent and dependent variations are available from polygons. It will help the design to see the variation and cost effects of the manufacturing variation at the same time in the early design stage. As the determination of tolerance sensitivities and the resultant variations for different nominal dimensions is convenient, such analysis in an iterative process at an early design stage becomes possible.

Due to the complexity of the assembly, more cases should be collected and tested for 2D and further 3D assembly analysis by the geometric and analytical approach. The geometric features in the tolerance analysis need to be found and summarized to increase the understanding of the assembly tolerance. The libraries of problems need to be built to test the geometric rules and guidelines.
REFERENCES


Chase, K. W., Gao, J and Magleby, S. D. "Generalized approach for 2-D assembly tolerance analysis of mechanical assembly with small kinematic adjustment," to be published, 1994


References 175


Wilson, R. J. (1985), "Introduction to Graph Theory," Longman, 1985


APPENDIX

A4.1 TAPEHUB VARIATIONS AT THE PREDICTED DIMENSIONS

![Graph showing variations in VRS, QV, and REAL for tapehub]

Figure A.1 Variations drl and dΔx for tapehub

Effect of Number of Variables in Design Dimension

The accuracy of the QV method is affected by the number of design dimensions. If only two independent tolerances are considered, the variations predicted by three methods are plotted in Figure A.2.

It means that if all tolerances are set to zero except two design parameters for tapehub, the variation values of QV are much closer to the real value. The bars in Figure A.2 can be compared with the bars in Figure A.1. In mechanical assemblies, more independent variables need to be considered as the variation sources. Therefore, VRS will be more appropriate estimation in the real design analysis.
A4.5 DATA FOR SUSPENSION SYSTEMS

Double Arm Suspension

Table A.1 Comparison of variation of scrub (du) by different methods

<table>
<thead>
<tr>
<th></th>
<th>VRS min.</th>
<th>QV min.</th>
<th>Real min.</th>
<th>VRS max</th>
<th>QV max</th>
<th>Real max</th>
</tr>
</thead>
<tbody>
<tr>
<td>estimation</td>
<td>0.01711696</td>
<td>0.003942241</td>
<td>0.018368</td>
<td>0.1561738</td>
<td>0.09883576</td>
<td>0.194799</td>
</tr>
<tr>
<td>real</td>
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<td>0.019002</td>
<td>0.018368</td>
<td>0.170373</td>
<td>0.156279</td>
<td>0.194799</td>
</tr>
<tr>
<td>rla</td>
<td>17</td>
<td>17</td>
<td>17</td>
<td>13</td>
<td>13</td>
<td>13</td>
</tr>
<tr>
<td>r2angle</td>
<td>269.8733°</td>
<td>276°</td>
<td>276°</td>
<td>256°</td>
<td>256°</td>
<td>256°</td>
</tr>
</tbody>
</table>

Table A.2 Comparison of scrub plus variation (u+du) by different methods

<table>
<thead>
<tr>
<th></th>
<th>VRS min.</th>
<th>QV min.</th>
<th>Real min.</th>
<th>VRS max</th>
<th>QV max</th>
<th>Real max</th>
</tr>
</thead>
<tbody>
<tr>
<td>estimation</td>
<td>0.04390597</td>
<td>0.05012825</td>
<td>0.02362347</td>
<td>1.634404</td>
<td>1.577206</td>
<td>1.708148</td>
</tr>
<tr>
<td>real</td>
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<td>0.048723</td>
<td>0.02362347</td>
<td>1.700719</td>
<td>1.70719</td>
<td>1.708148</td>
</tr>
<tr>
<td>rla</td>
<td>14.99105</td>
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<td>15.9416</td>
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<td>13</td>
<td>13</td>
</tr>
<tr>
<td>rf2</td>
<td>15.51174</td>
<td>15.51186</td>
<td>13.53</td>
<td>13.53</td>
<td>13.53</td>
<td>13.67131</td>
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<tr>
<td>angle</td>
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<td>265.9093</td>
<td>273.49334</td>
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<td>256</td>
<td>256</td>
</tr>
</tbody>
</table>
McPherson Strut Suspension

Table A.3 Comparison of variation of scrub (du) by different methods

<table>
<thead>
<tr>
<th></th>
<th>VRS min.</th>
<th>QV min.</th>
<th>Real min.</th>
<th>VRS max</th>
<th>QV max</th>
<th>Real max</th>
</tr>
</thead>
<tbody>
<tr>
<td>estimation</td>
<td>0.01806145</td>
<td>0.001707171</td>
<td>0.028726</td>
<td>0.2111070</td>
<td>0.133821</td>
<td>0.221431</td>
</tr>
<tr>
<td>real</td>
<td>0.30124</td>
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<td></td>
<td>0.220581</td>
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<td></td>
</tr>
<tr>
<td>rla</td>
<td>14.5</td>
<td>14.14173</td>
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<td>10.5</td>
<td>10.5</td>
<td>10.5</td>
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<td>26</td>
<td>26.30842</td>
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<td>26</td>
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<td>angle</td>
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<td>230</td>
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Table A.4 Comparison of scrub plus variation (u+du) by different methods

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<th>VRS min.</th>
<th>QV min.</th>
<th>Real min.</th>
<th>VRS max</th>
<th>QV max</th>
<th>Real max</th>
</tr>
</thead>
<tbody>
<tr>
<td>estimation</td>
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<td>0.05875889</td>
<td>2.175964</td>
<td>2.110851</td>
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<td>real</td>
<td>0.09691372</td>
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<td>rla</td>
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<td>12.23101</td>
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<td>27.82179</td>
<td>26.01825</td>
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<td>30</td>
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<td>angle</td>
<td>242.3196</td>
<td>242.3184</td>
<td>242.3401</td>
<td>230</td>
<td>230</td>
<td>230</td>
</tr>
</tbody>
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A5.1 DATA FOR ONE-WAY CLUTCH

Geometric Sensitivity Matrices

Table A.5 A Matrix (one-way clutch)

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>c</th>
<th>e</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
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<td>0.122187907</td>
<td>-0.061093954</td>
</tr>
<tr>
<td>Y</td>
<td>0.5</td>
<td>1.992506985</td>
<td>-0.496253493</td>
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<tr>
<td>0</td>
<td>0</td>
<td>0</td>
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</tbody>
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Table A.6 B Matrix (one-way clutch)

<table>
<thead>
<tr>
<th></th>
<th>φ1</th>
<th>φ2</th>
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</thead>
<tbody>
<tr>
<td>X</td>
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<tr>
<td>Y</td>
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<td>4.810537912</td>
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</tbody>
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Tolerance Sensitivity Matrix

Table A.7 S = -B⁻¹A Matrix (one-way clutch)

<table>
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<tr>
<th></th>
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<th>e</th>
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<tr>
<td>b</td>
<td>-4.061396118</td>
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<tr>
<td>φ1</td>
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<td>-0.414196296</td>
<td>0.10315967</td>
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<tr>
<td>φ2</td>
<td>-0.10393848</td>
<td>-0.414196296</td>
<td>0.10315967</td>
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</tbody>
</table>
A7.1 DATA FOR BICYCLE

Geometric Sensitivity Matrices

Table A.8 A Matrix (bicycle)

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>h</th>
<th>θ</th>
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</thead>
<tbody>
<tr>
<td>X</td>
<td>2.22E-16</td>
<td>2.22E-16</td>
<td>-1</td>
<td>0.069756</td>
<td>-0.069756</td>
<td>0.0697565</td>
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</tr>
<tr>
<td>Y</td>
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<td>-1</td>
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<td>0.997564</td>
<td>-0.997564</td>
<td>1</td>
<td>5.085188</td>
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<tr>
<td>θ</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Table A.9 B Matrix (bicycle)

<table>
<thead>
<tr>
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<th>φ</th>
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<th>i</th>
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</thead>
<tbody>
<tr>
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<tr>
<td>Y</td>
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<tr>
<td>θ</td>
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</tr>
</tbody>
</table>

Tolerance Sensitivity Matrix

Table A.10 S = -B⁻¹A Matrix (bicycle)

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>h</th>
<th>θ</th>
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<tbody>
<tr>
<td>φ</td>
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<td>0</td>
<td>0</td>
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</table>

A7.2 DATA FOR QUICK RETURN MECHANISM

Geometric Sensitivity Matrices

Table A.11 A Matrix (quick return mechanism)

<table>
<thead>
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</tr>
</thead>
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<tr>
<td>Y1</td>
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<tr>
<td>θ1</td>
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<td>0</td>
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<tr>
<td>X2</td>
<td>2.22E-16</td>
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<td>-0.707107</td>
</tr>
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<td>Y2</td>
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<td>0.7071068</td>
<td>-0.707107</td>
</tr>
<tr>
<td>θ2</td>
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<td>0</td>
</tr>
</tbody>
</table>
Table A.12 B Matrix (quick return mechanism)

<table>
<thead>
<tr>
<th></th>
<th>$\phi_1$</th>
<th>$\phi_2$</th>
<th>$u_1$</th>
<th>$\phi_3$</th>
<th>$\phi_4$</th>
<th>$u_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1$</td>
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<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$Y_1$</td>
<td>-1.75E-06</td>
<td>1.08253</td>
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<td>0</td>
</tr>
<tr>
<td>$\theta_1$</td>
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<td>1</td>
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<td>0</td>
<td>0</td>
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<tr>
<td>$X_2$</td>
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</tr>
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<td>1</td>
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</tbody>
</table>

Tolerance Sensitivity Matrix

Table A.13 $S = -B^{-1}A$ Matrix (quick return mechanism)

<table>
<thead>
<tr>
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<th>$h$</th>
<th>$a$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_1$</td>
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<td>-0.46188</td>
<td>-0.46188</td>
</tr>
<tr>
<td>$\phi_2$</td>
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<td>0.46188</td>
</tr>
<tr>
<td>$u_1$</td>
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<td>1.1547005</td>
<td>1.154701</td>
</tr>
<tr>
<td>$\phi_3$</td>
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<td>1.131375</td>
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<tr>
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</tr>
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A7.3 DATA FOR TAPEHUB

Geometric Sensitivity Matrices

Table A.14 A Matrix (tapehub)

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<th>$a$</th>
<th>$r$</th>
<th>$c$</th>
<th>$i$</th>
<th>$g$</th>
<th>$h$</th>
<th>$\theta$</th>
</tr>
</thead>
<tbody>
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</tr>
<tr>
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<td>-1</td>
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<td>$\theta$</td>
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</table>

Table A.15 B Matrix (tapehub)

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<th>$rl$</th>
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</thead>
<tbody>
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</tbody>
</table>
Tolerance Sensitivity Matrix

Table A.16 $S = -B^{-1}A$ Matrix (tapehub)

<table>
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<tr>
<th></th>
<th>b</th>
<th>a</th>
<th>r</th>
<th>e</th>
<th>i</th>
<th>g</th>
<th>h</th>
<th>θ</th>
</tr>
</thead>
<tbody>
<tr>
<td>u</td>
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<td>-2E-16</td>
<td>0.267949</td>
<td>-4.9E-32</td>
<td>-4.9E-32</td>
<td>1.035276</td>
<td>1.035276</td>
<td>-0.157931</td>
</tr>
<tr>
<td>φ</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>rl</td>
<td>-0.267949</td>
<td>1</td>
<td>1.035276</td>
<td>1</td>
<td>1</td>
<td>0.267949</td>
<td>0.267949</td>
<td>-0.378375</td>
</tr>
</tbody>
</table>

leff = 0.370375

A7.4 DATA FOR TRUSS

Table A.17 Variables and dimensions for truss

<table>
<thead>
<tr>
<th>Variable</th>
<th>mm</th>
<th>Variable</th>
<th>rad</th>
<th>deg</th>
</tr>
</thead>
<tbody>
<tr>
<td>L1</td>
<td>50</td>
<td>φ1</td>
<td>2.75183192</td>
<td>157.668355</td>
</tr>
<tr>
<td>L2</td>
<td>25</td>
<td>φ2</td>
<td>2.6681415</td>
<td>152.873247</td>
</tr>
<tr>
<td>L3</td>
<td>30</td>
<td>φ3</td>
<td>0.86321189</td>
<td>49.4583981</td>
</tr>
</tbody>
</table>

Table A.18 Other value for truss

<table>
<thead>
<tr>
<th>Angle</th>
<th>rad</th>
<th>deg</th>
</tr>
</thead>
<tbody>
<tr>
<td>angle A</td>
<td>0.38976073</td>
<td>22.331645</td>
</tr>
<tr>
<td>angle B</td>
<td>0.47345116</td>
<td>27.1267531</td>
</tr>
<tr>
<td>angle C</td>
<td>2.27838076</td>
<td>130.541602</td>
</tr>
</tbody>
</table>

Geometric Sensitivity Matrices

Table A.19 A Matrix (A start)

<table>
<thead>
<tr>
<th></th>
<th>L1</th>
<th>L2</th>
<th>L3</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>-0.925</td>
<td>0.65</td>
<td>1</td>
</tr>
<tr>
<td>Y</td>
<td>0.3799671</td>
<td>-0.7599342</td>
<td>-2.449E-16</td>
</tr>
<tr>
<td>θ</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table A.20 B Matrix (A start)

<table>
<thead>
<tr>
<th></th>
<th>φ1</th>
<th>φ2</th>
<th>φ3</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>7.3476E-15</td>
<td>18.9983552</td>
<td>7.3476E-15</td>
</tr>
<tr>
<td>Y</td>
<td>7.1054E-15</td>
<td>46.25</td>
<td>30</td>
</tr>
<tr>
<td>θ</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Table A.21 A Matrix (B start)

<table>
<thead>
<tr>
<th></th>
<th>L1</th>
<th>L2</th>
<th>L3</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>1</td>
<td>-0.89</td>
<td>-0.925</td>
</tr>
<tr>
<td>Y</td>
<td>-2.449E-16</td>
<td>0.45596052</td>
<td>-0.3799671</td>
</tr>
<tr>
<td>θ</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table A.22 B Matrix (B start)

<table>
<thead>
<tr>
<th></th>
<th>φ1</th>
<th>φ2</th>
<th>φ3</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>1.2246E-14</td>
<td>-1.881E-16</td>
<td>11.3990131</td>
</tr>
<tr>
<td>Y</td>
<td>50</td>
<td>-1.421E-14</td>
<td>22.25</td>
</tr>
<tr>
<td>θ</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Table A.23 A Matrix (C start)

<table>
<thead>
<tr>
<th></th>
<th>L1</th>
<th>L2</th>
<th>L3</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>-0.89</td>
<td>1</td>
<td>0.65</td>
</tr>
<tr>
<td>Y</td>
<td>-0.4559605</td>
<td>-2.449E-16</td>
<td>0.75993421</td>
</tr>
<tr>
<td>θ</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table A.24 B Matrix (C start)

<table>
<thead>
<tr>
<th></th>
<th>φ1</th>
<th>φ2</th>
<th>φ3</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>22.7980262</td>
<td>0</td>
<td>7.1054E-15</td>
</tr>
<tr>
<td>Y</td>
<td>-19.5</td>
<td>25</td>
<td>2.8422E-14</td>
</tr>
<tr>
<td>θ</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Tolerance Sensitivity Matrix

Table A.25 $S = -B^{-1}A$ Matrix (truss)

<table>
<thead>
<tr>
<th></th>
<th>L1</th>
<th>L2</th>
<th>L3</th>
</tr>
</thead>
<tbody>
<tr>
<td>φ1</td>
<td>0.03903847</td>
<td>-0.0438634</td>
<td>-0.0285112</td>
</tr>
<tr>
<td>φ2</td>
<td>0.04868843</td>
<td>-0.0342135</td>
<td>-0.0526361</td>
</tr>
<tr>
<td>φ3</td>
<td>-0.0877269</td>
<td>0.07807693</td>
<td>0.08114738</td>
</tr>
</tbody>
</table>

Table A.26 Pseudovectors (truss)

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>BA = L1</td>
<td>50</td>
</tr>
<tr>
<td>AC = L3</td>
<td>30</td>
</tr>
<tr>
<td>CB = L2</td>
<td>25</td>
</tr>
</tbody>
</table>
A7.5 SOME DERIVATION AND DATA FOR FOUR-BAR

The denominator of tolerance sensitivity for four-bar is the same. The magnitudes of the numerators are obtained from the following.

**Angle versus Length**

![Diagram](image_url)

Figure A.3 L3 column: Projection of pseudovectors AD, CA and DC onto L3

\[
AD_{L3} = AD \cos (\pi - \phi_4) = GD  \tag{A.1}
\]

\[
CA_{L3} = CA \cos \beta_2 = CG  \tag{A.2}
\]

\[
DC_{L3} = DC  \tag{A.3}
\]

![Diagram](image_url)

Figure A.4 L4 column: Projection of pseudovectors AD, CA and DC onto L4

\[
AD_{L4} = AD  \tag{A.4}
\]

\[
CA_{L4} = CA \cos (\beta_5 + \phi_1) = HA  \tag{A.5}
\]

\[
DC_{L4} = DC \cos (\pi - \phi_4) = DH  \tag{A.6}
\]

![Diagram](image_url)

Figure A.5 L1 column: Projection of pseudovectors AD, CA and DC onto L1
\[ AD_{L1} = AD \cos \phi_1 = AI \]  \hspace{1cm} (A.7)
\[ CA_{L1} = CA \cos \beta_5 = JA \]  \hspace{1cm} (A.8)
\[ DC_{L1} = DC \cos (\beta_3 - \phi_1) = IJ \]  \hspace{1cm} (A.9)

**Angle versus Angle**

The angle \( \phi_2 \) shares the common pseudovector \( AD \) with the dependent angle \( \phi_3 \).

The sensitivities are

\[
\begin{bmatrix}
\frac{\Delta \phi_3}{L2 \Delta \phi_2} + \frac{\Delta \phi_2}{L2 \Delta \phi_2} \\
\frac{\Delta \phi_4}{L2 \Delta \phi_2} \\
\frac{\Delta \phi_1}{L2 \Delta \phi_2}
\end{bmatrix}
= \begin{bmatrix}
\frac{AD_{L2} \, d\phi_2}{2 \, ADC} \\
\frac{CA_{L2} \, d\phi_2}{2 \, ADC} \\
-\frac{DC_{L2} \, d\phi_2}{2 \, ADC}
\end{bmatrix}
\]  \hspace{1cm} (A.10)

\[ AD_{L2} \, d\phi_2 = AD \sin (\phi_1 + \phi_2) = MK \]  \hspace{1cm} (A.11)
\[ CA_{L2} \, d\phi_2 = CA \sin (\phi_2 - \beta_5) = CM \]  \hspace{1cm} (A.12)
\[ DC_{L2} \, d\phi_2 = DC \sin (\phi_2 - \beta_5 + \beta_2) = KC \]  \hspace{1cm} (A.13)

The angular projection direction, shown in Figure 7.24, is positive in a direction corresponding to the increase of angle \( \phi_2 \). The projection of \( AD \) onto \( L2 \, d\phi_2 \) has the same direction as \( L2 \, d\phi_2 \), so the sensitivity would be positive. The projection of \( CA \) has the same direction as \( L2 \, \phi_2 \) and the numerator would also be the positive. \( DC \) has projection opposite the \( L2 \, \phi_2 \) direction, so a negative sign is assigned. The sensitivity relative to \( \phi_2 \) would be:

\[
\begin{bmatrix}
\Delta \phi_3 \\
\Delta \phi_2 \\
\Delta \phi_4 \\
\Delta \phi_2 \\
\Delta \phi_1 \\
\Delta \phi_2
\end{bmatrix}
= \begin{bmatrix}
-1 + L2 \cdot \frac{AD_{L2} \, d\phi_2}{2 \, ADC} \\
L2 \cdot \frac{CA_{L2} \, d\phi_2}{2 \, ADC} \\
-L2 \cdot \frac{DC_{L2} \, d\phi_2}{2 \, ADC}
\end{bmatrix}
\]  \hspace{1cm} (A.14)
Table A.27 Variables and dimensions for four-bar

<table>
<thead>
<tr>
<th>Variable</th>
<th>mm</th>
<th>Variable</th>
<th>rad</th>
<th>deg</th>
</tr>
</thead>
<tbody>
<tr>
<td>L1</td>
<td>50</td>
<td>φ1</td>
<td>1.4014274</td>
<td>80.29587</td>
</tr>
<tr>
<td>L2</td>
<td>25</td>
<td>φ2</td>
<td>1.3613568</td>
<td>78</td>
</tr>
<tr>
<td>L3</td>
<td>60</td>
<td>φ3</td>
<td>1.727784</td>
<td>98.99473</td>
</tr>
<tr>
<td>L4</td>
<td>28</td>
<td>φ4</td>
<td>1.7926171</td>
<td>102.7094</td>
</tr>
</tbody>
</table>

Geometric Sensitivity Matrices

Table A.28 A Matrix (four-bar)

<table>
<thead>
<tr>
<th></th>
<th>L2</th>
<th>L3</th>
<th>L4</th>
<th>L1</th>
<th>φ2</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>0.207912</td>
<td>-0.998625</td>
<td>0.16856</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Y</td>
<td>0.978148</td>
<td>0.052428</td>
<td>-0.985691</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Table A.29 B Matrix (four-bar)

<table>
<thead>
<tr>
<th></th>
<th>φ3</th>
<th>φ4</th>
<th>φ1</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>24.45369</td>
<td>27.599357</td>
<td>0</td>
</tr>
<tr>
<td>Y</td>
<td>-5.1977923</td>
<td>54.719691</td>
<td>50</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Tolerance Sensitivity Matrix

Table A.30 $S = -B^{-1}A$ Matrix (four-bar)

<table>
<thead>
<tr>
<th></th>
<th>L2</th>
<th>L3</th>
<th>L4</th>
<th>L1</th>
<th>φ2</th>
</tr>
</thead>
<tbody>
<tr>
<td>φ3</td>
<td>0.015874</td>
<td>0.003759</td>
<td>-0.017085</td>
<td>-0.00288</td>
<td>-0.842041</td>
</tr>
<tr>
<td>φ4</td>
<td>-0.021598</td>
<td>0.032852</td>
<td>0.009031</td>
<td>-0.033681</td>
<td>0.746068</td>
</tr>
<tr>
<td>φ1</td>
<td>0.005724</td>
<td>-0.03661</td>
<td>0.008055</td>
<td>0.036561</td>
<td>-0.904027</td>
</tr>
</tbody>
</table>

Table A.31 Pseudovectors (four-bar)

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>CA</td>
<td>60.372007</td>
</tr>
<tr>
<td>AD = L4</td>
<td>28</td>
</tr>
<tr>
<td>DC = L3</td>
<td>60</td>
</tr>
</tbody>
</table>

CA = 60.372007
## A7.6 DATA FOR REMOTE POSITIONING MECHANISM

### Geometric Sensitivity Matrices

**Table A.32 A Matrix (remote positioning mechanism)**

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
<th>i</th>
</tr>
</thead>
<tbody>
<tr>
<td>X1</td>
<td>-1</td>
<td>-0.503774</td>
<td>1</td>
<td>0.503774</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Y1</td>
<td>1E-16</td>
<td>-0.863836</td>
<td>-2.4E-16</td>
<td>0.863836</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>θ1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

|    |        |            |      |     |      | 6.1E-17 | -0.735553 | -2E-16 | 0.735553 |
| X2 | 0      | 0          | 0    | 0   | 0    | 0   | 0   | 0   |
| Y2 | 0      | 0          | 0    | 0   | -1   | -0.677467 | 1   | 0.677467 |
| θ2 | 0      | 0          | 0    | 0   | 0    | 0   | 0   | 0   |

**Table A.33 B Matrix (remote positioning mechanism)**

<table>
<thead>
<tr>
<th></th>
<th>φ1</th>
<th>φ2</th>
<th>φ3</th>
<th>φ4</th>
<th>φ5</th>
<th>φ6</th>
</tr>
</thead>
<tbody>
<tr>
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<td>-8.983889</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Y1</td>
<td>22</td>
<td>27.239249</td>
<td>5.239249</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>θ1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>X2</th>
<th>Y2</th>
<th>θ2</th>
</tr>
</thead>
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<tr>
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</tr>
<tr>
<td></td>
<td>0</td>
<td>5.329E-15</td>
<td>49.3</td>
</tr>
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<td></td>
<td>0</td>
<td>5.329E-15</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-1</td>
<td>-1</td>
</tr>
</tbody>
</table>

### Tolerance Sensitivity Matrix

**Table A.34 S = -B⁻¹A Matrix (remote positioning mechanism)**

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
<th>i</th>
</tr>
</thead>
<tbody>
<tr>
<td>φ1</td>
<td>0.11131</td>
<td>0.0560753</td>
<td>-0.11131</td>
<td>-0.056075</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>φ2</td>
<td>-0.084802</td>
<td>-0.003456</td>
<td>0.0848021</td>
<td>0.003456</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>φ3</td>
<td>-0.026508</td>
<td>-0.052619</td>
<td>0.0265083</td>
<td>0.052619</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>φ4</td>
<td>0.084802</td>
<td>0.0034558</td>
<td>-0.084802</td>
<td>-0.003456</td>
<td>0.105389</td>
<td>0.0713977</td>
<td>-0.105389</td>
<td>-0.071398</td>
</tr>
<tr>
<td>φ5</td>
<td>-0.084802</td>
<td>-0.003456</td>
<td>0.0848021</td>
<td>0.003456</td>
<td>-0.086707</td>
<td>-0.043821</td>
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<td>0.0438212</td>
</tr>
<tr>
<td>φ6</td>
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<td>0.0034558</td>
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<td>-0.003456</td>
<td>-0.018682</td>
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<td>0.0275765</td>
</tr>
</tbody>
</table>
### Geometric Sensitivity Matrix

Table A.35  A Matrix (remote positioning mechanism including angles)

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
<th>i</th>
<th>θ2</th>
<th>θ3</th>
<th>θ4</th>
</tr>
</thead>
<tbody>
<tr>
<td>X1</td>
<td>-1</td>
<td>-0.503774</td>
<td>1</td>
<td>0.503774</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Y1</td>
<td>1E-16</td>
<td>-0.863836</td>
<td>0</td>
<td>0.863836</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>θ1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>X2</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-0.735553</td>
<td>0</td>
<td>0.735553</td>
</tr>
<tr>
<td>Y2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0.677467</td>
<td>1</td>
<td>0.677467</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>θ2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
</tbody>
</table>

### Tolerance Sensitivity Matrix

Table A.36  S = -B⁻¹A Matrix (remote positioning mechanism including angles)

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
<th>i</th>
<th>θ2</th>
<th>θ3</th>
<th>θ4</th>
</tr>
</thead>
<tbody>
<tr>
<td>φ1</td>
<td>0.1113</td>
<td>0.05608</td>
<td>-0.11131</td>
<td>-0.0561</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>φ2</td>
<td>-0.0848</td>
<td>-0.00346</td>
<td>0.0848</td>
<td>0.00346</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>φ3</td>
<td>-0.0265</td>
<td>-0.05262</td>
<td>0.02651</td>
<td>0.05262</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>φ4</td>
<td>0.0848</td>
<td>0.00346</td>
<td>-0.0848</td>
<td>-0.0035</td>
<td>0.10539</td>
<td>0.0714</td>
<td>-0.1054</td>
<td>-0.0714</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>φ5</td>
<td>-0.0848</td>
<td>-0.00346</td>
<td>0.0848</td>
<td>0.00346</td>
<td>-0.0867</td>
<td>-0.04382</td>
<td>0.0867</td>
<td>0.04382</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>φ6</td>
<td>0.0848</td>
<td>0.00346</td>
<td>-0.0848</td>
<td>-0.0035</td>
<td>-0.0187</td>
<td>-0.02758</td>
<td>0.0187</td>
<td>0.02758</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
</tbody>
</table>

### Table A.37  Pseudovectors (remote positioning mechanism)

| DC=-c | 22 | GF=g | 49.3 |
| BD  | 19.016651 | FG  | 41.655768 |
| CB=b | 10.4 | FE=t | 12.9 |

### A7.7 DATA FOR RATCHET AND PAWL

#### Geometric Sensitivity Matrices

Table A.38  A Matrix (ratchet and pawl)

<table>
<thead>
<tr>
<th></th>
<th>xp</th>
<th>yp</th>
<th>dp</th>
<th>dr</th>
<th>dt1</th>
<th>dt2</th>
<th>angro</th>
</tr>
</thead>
<tbody>
<tr>
<td>X1</td>
<td>1</td>
<td>0</td>
<td>-0.959771</td>
<td>-0.243826</td>
<td>-0.969819</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Y1</td>
<td>0</td>
<td>1</td>
<td>0.280785</td>
<td>-0.969819</td>
<td>0.243826</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>θ1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>X2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Y2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.243826</td>
<td>0.970769</td>
<td>0</td>
</tr>
<tr>
<td>θ2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
</tr>
</tbody>
</table>
Table A.39  B Matrix (ratchet and pawl)

<table>
<thead>
<tr>
<th></th>
<th>( \Phi_p )</th>
<th>( \Phi_r )</th>
<th>( \Phi_{\tau 2} )</th>
<th>( \Phi_r )</th>
<th>ds1</th>
<th>ds2</th>
</tr>
</thead>
<tbody>
<tr>
<td>X1</td>
<td>0.3</td>
<td>0.384236</td>
<td>0</td>
<td>0</td>
<td>-0.243826</td>
<td>0</td>
</tr>
<tr>
<td>Y1</td>
<td>0</td>
<td>0.287931</td>
<td>0</td>
<td>0.4</td>
<td>-0.969819</td>
<td>0</td>
</tr>
<tr>
<td>( \theta_1 )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>X2</td>
<td>0</td>
<td>0</td>
<td>0.0205534</td>
<td>0</td>
<td>-0.243826</td>
<td>0.970769</td>
</tr>
<tr>
<td>Y2</td>
<td>0</td>
<td>0</td>
<td>-0.020634</td>
<td>0</td>
<td>-0.969819</td>
<td>0.240017</td>
</tr>
<tr>
<td>( \theta_2 )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Tolerance Sensitivity Matrix

Table A.40  \( S = -B^{-1}A \) Matrix (ratchet and pawl)

<table>
<thead>
<tr>
<th></th>
<th>( x_p )</th>
<th>( y_p )</th>
<th>( d_p )</th>
<th>( d_r )</th>
<th>( d_{t1} )</th>
<th>( d_{t2} )</th>
<th>angro</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Phi_p )</td>
<td>0.933334</td>
<td>3.2000008</td>
<td>0.0027255</td>
<td>-3.330993</td>
<td>-1.896044</td>
<td>-3.772583</td>
<td>0.0941795</td>
</tr>
<tr>
<td>( \Phi_r )</td>
<td>-3.331291</td>
<td>-2.498468</td>
<td>2.4957427</td>
<td>3.235316</td>
<td>4.341807</td>
<td>3.6642223</td>
<td>-0.091474</td>
</tr>
<tr>
<td>( \Phi_{\tau 2} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( \Phi_r )</td>
<td>2.397956</td>
<td>-0.701533</td>
<td>-2.498468</td>
<td>0.095677</td>
<td>-2.445764</td>
<td>0.1083604</td>
<td>-0.002705</td>
</tr>
<tr>
<td>ds1</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0.531709</td>
<td>1.1325701</td>
<td>-0.028274</td>
</tr>
<tr>
<td>ds2</td>
<td>0</td>
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<td>0</td>
<td>0</td>
<td>1.13257</td>
<td>0.5317094</td>
<td>-0.028274</td>
</tr>
</tbody>
</table>

Table A.41  Pseudovectors (ratchet and pawl)

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>PC</td>
<td>0.4002454</td>
</tr>
<tr>
<td>BF</td>
<td>0.5</td>
</tr>
<tr>
<td>CB</td>
<td>0.3</td>
</tr>
</tbody>
</table>
NEW TOLERANCE ANALYSIS METHODS
FOR PRELIMINARY DESIGN IN MECHANICAL ASSEMBLIES

A Dissertation
Presented to the
Department of Mechanical Engineering
Brigham Young University

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

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by
Hanqi Huo
December 1995
NEW TOLERANCE ANALYSIS METHODS
FOR PRELIMINARY DESIGN IN MECHANICAL ASSEMBLIES

Hanqi Huo
Department of Mechanical Engineering
Ph.D. Degree, December, 1995

Abstract

New methods have been developed to evaluate tolerance sensitivity and variations in mechanical assemblies. The methods are applicable to both detailed and preliminary design.

The variation polygon and frame polygon provide new graphical representations for determining the sensitivity of assemblies to variation. Tolerance sensitivities can be derived in closed form from vector variation geometry. The symbolic and graphical natures of these tools provide considerably greater insight into the sources and effects of variation than simple numerical values of the sensitivities. They help to identify and develop an understanding of the mechanism governing tolerance sensitivities and assembly variations.

The Variation Response Surface method combines design of experiments and response surface methodology with statistical tolerance analysis. It permits the evaluation of manufacturability in the early design stage. A multi-dimensional design space is created, with one axis for each component dimension. Assembly functions are used to calculate critical assembly features at each experimental point by linear and nonlinear analysis. Response surface for mean, variation or sensitivity may be fit. Surfaces may then be searched with optimization techniques to find the regions of minimum variation, or which are robust to variation. Two competing designs may also be evaluated by comparing variation response surfaces. A more accurate surface method, based on nonlinear optimization, was developed as an evaluation tool. It does not use response surface, but solve the nonlinear system repeatedly, so it is more costly.

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Parkinson, A. R. Committee Member
Sorensen, Carl D. Committee Member
Webb, Brent W. Graduate Coordinator
This dissertation, by Hanqi Huo, is accepted in its present form by the Department of Mechanical Engineering of Brigham Young University as satisfying the dissertation requirement for the degree of Doctor of Philosophy.

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- Skilled in computer-aided design and analysis
- Skilled in machine design and analysis
- Experienced in the application of optimization methods in design
- Experienced in dynamic control systems and measurement systems
- Proficient in assembly tolerance analysis and variation analysis
- Skilled in statistical design of experiments and quality control
- Experienced in modeling and analyzing engineering problems
- Capable of developing application software in FORTRAN, C, C++
- Experienced in teaching at the university level
- Excellent in Chinese language

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Thesis "CAD Optimization of Hydraulic Profiling Cutter"

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Northwestern Polytechnical University, Xian, CHINA

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- Advanced Mechanisms
- Advanced Vibration Analysis
- Computer-Aided Engineering Software Design
- Specific Topics in Dynamic System Modeling
- Computer-Aided Geometric Design
- Design of Control Systems
- Advanced Automatic Control Applications
- Stress Analysis and Design of Mechanical Structures
- Optimization Techniques in Engineering
- Expert System Design

AWARDS:
- Mechanical Engineering Scholarship - Brigham Young University
- Academic Honor Roll - Brigham Young University
WORK EXPERIENCE:

90-present  **Research and Teaching Assistant,**
Brigham Young University, Provo, Utah

- Developed a new tool - variation polygon for assembly tolerance analysis, which changes the analysis to a relationship-oriented approach and provides fast feedback and a visual tool for variation analysis in mechanical assembly.
- Developed variation surface approach to evaluate manufacturability of the design from tolerance view. Built dynamic model of control valve for industrial applications

- Assisted in teaching the following courses in Mechanical Engineering Department, Mathematics Department, and Honors Department:
  - Stress Analysis and Design
  - CAE and Programming
  - Linear Algebra
  - Honors Calculus

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- Taught the following courses: Design of Machine Tools, Hydraulic System of Machine Tools, Management and Organization of Industrial Maintenance, to senior students in mechanical and manufacturing engineering, and corporate managers; assisted in teaching of Control System, Metal Cutting and other engineering courses; directed senior design program for mechanical engineering students

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Research and design projects included:

- Optimization of spindle for a turning machine
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- Design of tape twisting machine
- Cutting dynamics

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CONFERENCES: