Characterization of Assembly Variation Analysis Methods

A Thesis

Presented to the

Department of Mechanical Engineering

Brigham Young University

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

By

Robert Cvetko

December 1997

This thesis by Robert Cvetko is accepted in its present form by the Department of Mechanical Engineering of Brigham Young University as satisfying the thesis requirement for the degree of Master of Science.

Kenneth W. Chase, Committee Chair

Carl D. Sorensen, Committee Member

Spencer P. Magleby, Committee Member

Date

Craig Smith, Graduate Coordinator

Characterization of Assembly

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ABSTRACT

This thesis is a study and comparison of current assembly variation analysis methods. New metrics are presented for gauging the accuracy of , not only the analysis methods, but also the problem information without knowing the exact answer. A hybrid method using Monte Carlo with the method of system moments is also presented and evaluated. Matching the accuracy of the problem and the analysis method allows for simple models and an estimate of confidence for the results. A framework for choosing the best method for different analysis situations summarizes the findings.

COMITTEE APPROVAL:

Kenneth Chase, Committee Chair

Carl D. Sorensen, Committee Member

Spencer P. Magleby, Committee Member

Date

Craig Smith, Graduate Coordinator

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Chapter 1. Introduction

Variation analysis, or tolerance analysis, is a valuable tool to design products for high quality and low cost. Variation analysis is used to predict the variation in assemblies due to the process variations of component dimensions and other features. There are several methods available to perform assembly variation analysis. The main methods are listed below.

- **Monte Carlo Simulation**: Approximate the output distribution by sampling the assembly function many times (thousands or more depending on desired certainty level) with random input values.
- **Method of System Moments**: Approximate the first four output distribution moments by combining the input variable moments and the sensitivities (derivatives) of the assembly function.
- **RSS First Order Approximation**: Approximate the assembly variation as the sum-squared product of each input variable standard deviation and its linear sensitivity.
- **Design of Experiments**: Approximate the total assembly variation and the contribution from each input variable by sampling the assembly function at specific combinations of the input variables.

Section 1.1. Statement of the Problem

Although there are several methods available for variation analysis, many engineers do not understand the strengths and weaknesses of each. The accuracy of an analysis is seldom estimated. Even considering the accuracy of the input information is not generally done. It is common for an engineer to learn and embrace only one of the methods, using it in all variation analysis situations.

The different methods are based on different assumptions, do not always yield the same results, and offer different kinds of information. Because different kinds of information are needed at different stages of product development and production, all of the variation analysis methods should be learned, and used where appropriate. The methods can even be used together to increase efficiency. Therefore the objectives of this thesis are:

- Compare the existing variation analysis methods to determine their strengths and weaknesses.
- Identify ways to estimate the accuracy of the methods without having to perform a full Monte Carlo simulation to benchmark.
- Present new hybrid methods to increase efficiency.
- Present a framework for selecting the optimum analysis method.

Section 1.2. Analysis and Review of Previous Work in the Field

The first part of this thesis (Chapter 2 through Chapter 6) demonstrates the state-of-the-art in variation analysis, and builds a foundation for the contributions. These chapters focus on the following:

- Explaining the background for several variation analysis methods
- Presenting current techniques to estimate the accuracy of the methods
- Demonstrating the methods on a sample problem to allow for comparison of accuracy

Because several variation analysis methods are being evaluated, the complete state-of-the-art for each method can not be presented. However, the state-of-the-art in variation analysis relative to the contributions of this thesis will be presented.

Section 1.3. Contributions to be Made by this Thesis

The rest of the chapters, Chapter 8 through Chapter 14, focus on the contributions made by this thesis. The main contributions are:

- Presenting and demonstrating new metrics for estimating the accuracy of the different variation analysis methods (Chapter 8 through Chapter 12)
- Presenting and evaluating a hybrid method that combines the strengths of several different analysis methods (Chapter 9 through Chapter 11)
- Presenting a framework for choosing the level of accuracy, and the method, for variation analysis (Chapter 13 and Chapter 14)

The value of these contributions is that with them an engineer can compare analysis needs to available analysis methods and select the optimum method, or even use the methods together as a hybrid. The accuracy of the analysis can be matched to the accuracy of the input information, allowing for efficiency, model simplicity, and an understanding of the confidence of the results.

One of the problems stated earlier in this thesis is that many engineers do not understand the various assembly variation analysis methods but use only one in all circumstances. Thus, even the first several chapters of this thesis may help to correct that problem by demonstrating each of the methods all in one place. The value that these contributions make is helping the variation analysis process become more efficient and useful in many different situations.

Section 1.4. Delimitations of the problem

This thesis will not be an exhaustive study of any one of the assembly variation analysis methods, as all of the methods are covered and compared. The test problem, presented in Chapter 2, was chosen because it is known to be quite non-linear and non-quadratic.

Additionally, there will be no experimental verification of the results of this thesis. The results will be compared to the Monte Carlo solution for a very large sample of simulated assemblies.

Section 1.5. Definition of General Terms and symbols

The following terms are presented here for reference. They will be referred to throughout the thesis.

$\mu_1 = \frac{1}{n}\sum_{i=1}^{n} x_i$	1st moment (or mean) of a statistical distribution
$\mu_2 = \sigma^2 = \frac{1}{n}\sum_{i=1}^{n} (x_i - \mu_1)^2$	2nd moment (or variance) of a statistical distribution
$\mu_3 = \frac{1}{n}\sum_{i=1}^{n} (x_i - \mu_1)^3$	3rd moment (or skewness) of a statistical distribution
$\mu_4 = \frac{1}{n}\sum_{i=1}^{n} (x_i - \mu_1)^4$	4th moment (or kurtosis) of a statistical distribution
$\alpha_3 = \mu_3 / \sigma^3$	Standardized skewness
$\alpha_4 = \mu_4 / \sigma^4$	Standardized kurtosis

Chapter 2. Setting Up an Assembly Variation Analysis Problem

This chapter defines what an assembly problem is, and the value of variation analysis in bringing together the engineering and manufacturing requirements. The engineering requirements are the limits on fit and function critical to assembly performance. These assembly specifications are applied to the dependent dimensions (or dependent variables) in the assembly. The values and variations of the part dimensions (or independent variables) determine the value and variation of the dependent variables.

The manufacturing requirements are the limits to size, shape, and location of part features. These requirements are normally applied to the assembly components in terms of tolerances. The variations in the manufacturing processes to make the parts cause the variations of the independent variables, and thus indirectly cause the variations in the dependent variables of the assembly. The difficulty lies in tying together the process variations and the assembly variations, in hopes of selecting the best processes to meet the engineering requirements most efficiently.

Assembly variation analysis is the tool to bring together the engineering requirements and the manufacturing capabilities, allowing the manufacturing processes and requirements to be chosen. Before the analysis can be performed, the objective for the analysis must be determined and the assembly function defined.

Section 2.1. Defining the Variation Analysis Objective

Defining the variation analysis objective is an important step in analyzing the variation in an assembly. The analysis objective defines the goals or reasons for performing the variation analysis. The analysis objective not only defines the characteristic of the assembly that is critical for performance, but it should also specify the desired output information. Once the analysis objective is clear, the variables that affect the objective

should be identified and their variation defined. The list below illustrates different types of possible output information from a variation analysis.

- Estimating the quality levels expected in production (upper, lower, or total rejects)
- Determining the manufacturing requirements and processes for the parts in the assembly to meet the engineering requirements
- Determining the sensitivity of the objective function to the independent variables to compare different designs or parameter values (robust design)
- Determining the percent contribution of each of the input variations to the assembly variation, to help focus improvement efforts
- Weighing the savings in quality loss against the cost of investments or changes

Once the problem objective is understood, the assembly function can be determined that will help reach the analysis objective.

Section 2.2. Defining an Assembly Function

In order to analyze assembly variations, a relationship between the independent and dependent variables must be found. This relationship can be found through experimentation, an explicit function, or a set of implicit equations. Experimentation means physically building the parts of an assembly and measuring the desired assembly characteristics. Experimentation is a valuable tool when relationships are unknown, like the effect of environmental noise.

If the assembly function is simple enough, the objective function can be expressed as an explicit equation in which the assembly objective (dependent variable) can be determined from the independent variables. More complicated assembly functions are not easily reducible to an explicit form. In these cases the assembly dimensions (dependent and

independent) are all incorporated into implicit equations representing closure in each of the necessary translational and rotational directions. Any assembly can be represented as a set of implicit equations, but an iterative process is required to solve them. Examples of explicit and implicit equations are presented in the next section (see Eq. 2-1 and Eq. 2-2). For the analysis in this thesis, the assembly function that will be used is the one-way clutch.

Section 2.3. One-way Clutch Model

Figure 2-1 below shows a one-way clutch assembly. The assembly consists of four different parts: one hub, one ring, four rollers, and four springs. The springs push the rollers out to remain in contact with both the ring and the hub. If the hub is turned counter-clockwise the rollers bind, causing the ring to turn with the hub. Turning the hub clockwise causes the rollers to slip and prevents the transmission of torque to the ring.

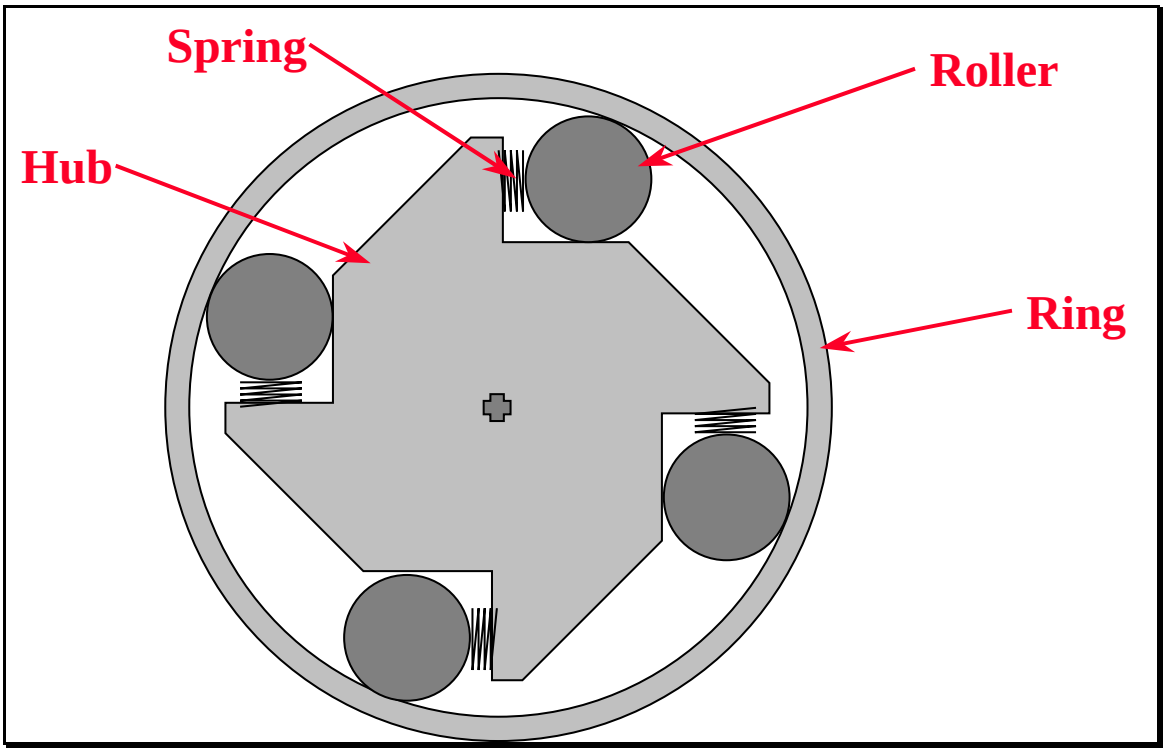


Figure 2-1: Drawing of the One-way Clutch Assembly

The binding of the rollers causes the transmission of torque between the hub and ring. The angle of contact between the roller and the ring is very important for the one-way clutch to function properly. The angle of contact and the dimensions that affect it are shown below in Figure 2-2.

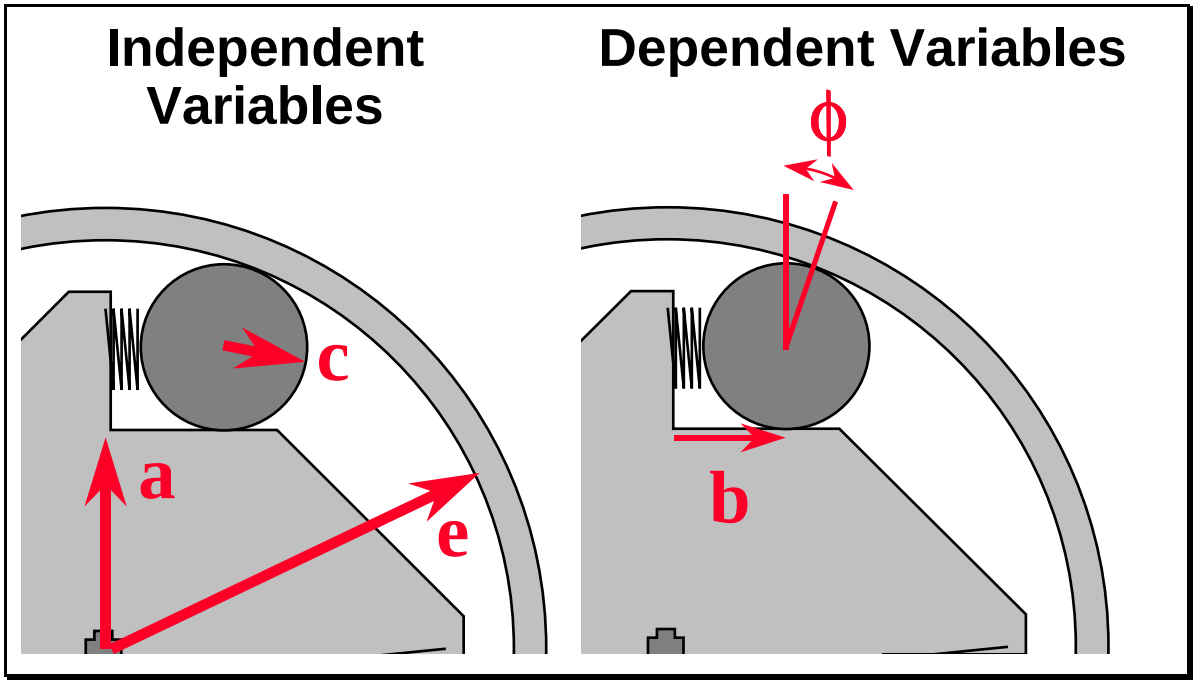


Figure 2-2: Important Dimensions for Clutch

The contact angle (ϕ) is the dependent variable in the assembly and determines the performance of the clutch. The three independent dimensions affect the value of the contact angle and the variable b (location of contact between the roller and the hub). The contact angle for the roller, when all of the independent variables are at their mean values, is 7.0184 degrees (nominal angle). For proper performance, the upper and lower specification limits for ϕ are based upon this nominal angle and ± 0.60 degree design limits. Table 2-1 below summarizes these values.

Table 2-1: Contact Angle Specifications

Contact Angle	Value (degrees)
Upper Limit	7.6184
Nominal Angle	7.0184
Lower Limit	6.4184

If the contact angle is outside of the limits, the clutch does not perform correctly and must be reworked, or scrapped. The cost of this rework, or the quality loss, is $20. This loss due to poor quality should be minimized. Variation analysis can help to predict the quality level of the clutches or, inversely, to determine the variations for the independent dimensions that will yield the desired quality level.

Dimensions a, c, and e are independent variables, which affect the contact angle ϕ. The variation in the contact angle is a function of the variations of the independent variables as well as the assembly function. The objective of this variation analysis problem is to determine the variation of the contact angle relative to its performance. Table 2-2 below shows the three independent variables, or dimensions, and the standard deviation of each. Each of the independent variables is assumed to be a statistically independent (not correlated with each other) and normally distributed random variable.

Table 2-2: Independent Dimensions for the Clutch

Variable	Mean	Standard Deviation
a - hub radius	27.645 mm	0.01666 mm
c - roller radius	11.430 mm	0.00333 mm
e - ring radius	50.800 mm	0.00416 mm

The contact angle for the one-way clutch is simple enough to be represented by the explicit function Eq. 2-1 (refer to Figure 2-2 for an explanation of the symbols).

$$\phi = \arccos\left(\frac{a+c}{e-c}\right) \qquad \textbf{Eq. 2-1}$$

Most assemblies are too complicated to be reduced into an explicit equation. Thus, a vector loop method can be used to find implicit equations [Chase 1995, 267]. Implicit equations can always be found for an assembly using the following vector loop method.

Section 2.4. Vector Loop Model and Assembly Function for the Clutch

The vector loop method uses the assembly drawing as the starting point. Vectors are drawn from part-to-part in the assembly, passing through the points of contact. The vectors represent the independent and dependent dimensions (variables) in the assembly. Figure 2-1 below shows the resulting vector loop for the clutch assembly.

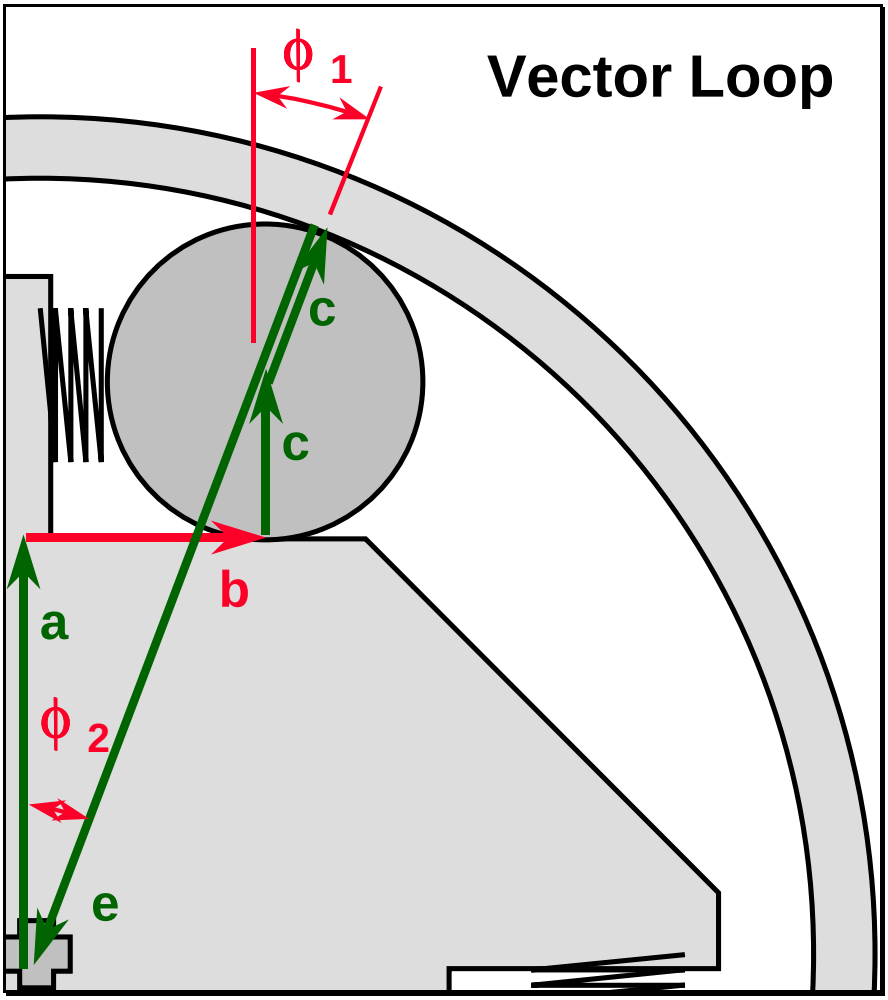


Figure 2-1: Vector Loop for the Clutch Assembly

The vectors pass through the points of contact between the three parts in the assembly. The vector b and the angles ϕ_1 and ϕ_2 are dependent variables. In this case it easy to see that $\phi_2 = \phi_1$ (the contact angle). Once the vector loop is defined, the implicit equations for the assembly can easily be extracted. Eq. 2-1 shows the set of implicit equations for the clutch assembly derived from the vectors.

$$\begin{aligned} h_x &= 0 = b + c\sin(\phi_1) - e\sin(\phi_1) \\ h_y &= 0 = a + c + c\cos(\phi_1) - e\cos(\phi_1) \\ h_\theta &= 0 = 90 - 90 + 90 - \phi_1 - 180 + \phi_2 + 90 \\ &= -\phi_1 + \phi_2 \end{aligned} \qquad \textbf{Eq. 2-1}$$

From the implicit equations it is also easy to see that ϕ_1 and ϕ_2 are equal. Therefore, the implicit equations can be simplified into just two equations for two unknowns as shown below by Eq. 2-2.

$$\begin{aligned} h_x &= 0 = b + c\sin(\phi) - e\sin(\phi) \\ h_y &= 0 = a + c + c\cos(\phi) - e\cos(\phi) \end{aligned} \qquad \textbf{Eq. 2-2}$$

Each of these assembly equations for the clutch equals zero because the vector loop is closed, and must maintain closure for the parts to assemble. However, if a vector loop is not closed (open loop vector to measure a gap for example) the equations equal the value of the opening (gap length in the h_y direction for example). In this case the implicit equation h_y contains only one dependent variable (the contact angle) and can be simplified down to the explicit equation for the contact angle shown above by Eq. 2-1. But most assemblies are too complex to easily derive an explicit equation for the desired assembly function.

Section 2.5. Summary for the Variation Analysis Problem

Defining the problem well consists of determining the analysis objectives and sources of variation and defining the assembly function. The one-way clutch assembly will be

analyzed with all of the analysis methods. The example problem, the one-way clutch assembly, was chosen for several different reasons:

- The clutch can be represented with a small number of variables (three independent and two dependent variables) for easy demonstration and comprehension.
- The clutch assembly is known to be quite non-linear, and even shown to be very non-quadratic [Glancy 1994, 43–45]. Therefore the output distribution for the contact angle is skewed even though the inputs are normally distributed.
- The contact angle for the clutch assembly can be represented explicitly, and not just implicitly to allow for use with any of the analysis methods.

Now that the sample analysis problem is defined, the different variation analysis method can be explained and demonstrated. The first method demonstrated will be Monte Carlo as it is widely used as a benchmark for the accuracy of the other methods. The following will be evaluated because they are the ones commonly used for assembly variation analysis:

- Chapter 3. Monte Carlo Simulation
- Chapter 4. Method of System Moments
- Chapter 5. RSS First-order Approximation
- Chapter 6. Quality Loss and Design of Experiments

Presenting the state-of-the-art with these methods, and demonstrating their use on the clutch sample problem will provide a foundation for understanding the value of contribution of this thesis.

Chapter 3. Monte Carlo Simulation

This chapter presents the Monte Carlo simulation method and the state-of-the-art for estimating its accuracy. This state-of-the-art on Monte Carlo will be used as the foundation for the contributions made in Chapter 9.

Monte Carlo Simulation is a method of choosing inputs from the probability distributions of randomly varying components, calculating the outputs, and analyzing the output distribution. In order to understand Monte Carlo simulation, a custom simulation program has been written and used along with the *Excel* add-in program *Crystal Ball*.

Section 3.1. Background on Monte Carlo

The Monte Carlo Method for variation analysis consists of setting the independent variables to random values according to their respective probability distributions. Figure 3-1 below graphically shows how the process works. The output distribution is a function of the distributions of the input variables and the assembly function. Thousands of samples of the input variables are combined to get a reliable measure of the output distribution.

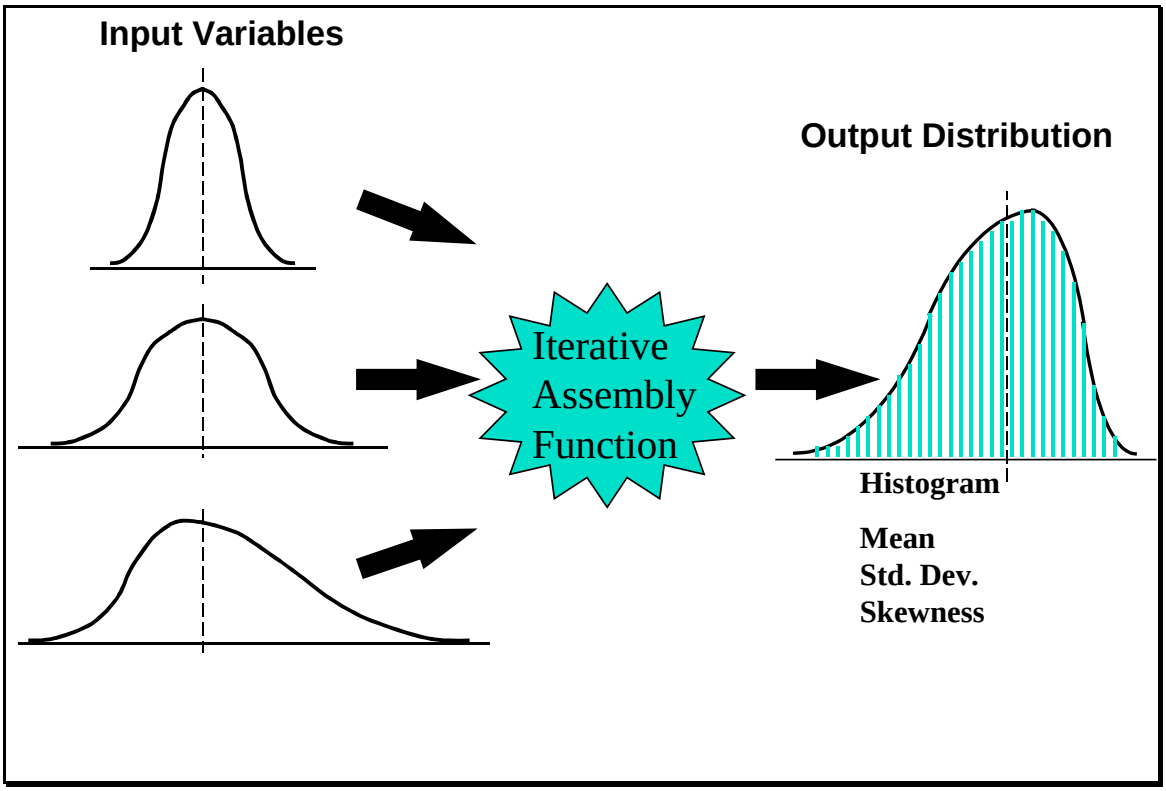


Figure 3-1: Graphical Representation of Monte Carlo Method

The Monte Carlo method consists of selecting random values for the independent variables, calculating the assembly function (an iterative process if the function is implicit), and analyzing the output. Sometimes hundreds of thousands of samples are required for accurate results.

The histogram of the outputs for the assembly function can be plotted. The rejects (assemblies that fall outside the specification limits) can be counted during the simulation, if the limits are known before the simulation. Or a distribution can be fit to the moments or percentiles of the Monte Carlo output and the rejects estimated (see Chapter 7 for using the Lambda distribution).

The number of samples required depends on the desired accuracy of the output. Section 3.5 will present the state-of-the-art in estimating the accuracy. Chapter 9 will contribute additional material for estimating the effectiveness of Monte Carlo simulation.

Section 3.2. Random Number Generation

One important aspect of a Monte Carlo simulation is the generation of the random numbers (or uniform random deviate) for the independent variables. The first step is generating uniform random deviate. Then they can be mapped onto the probability density functions of the independent variables, such as the normal distribution. This mapping can be done through the equations for the probability density functions if they are known, or by matching a Lambda distribution to data, discussed in Chapter 7.

To generate the uniform random deviate, the algorithm *ran2* [Press 1992, 282] was chosen because it has a period greater than 2×10^{18} random numbers before it repeats. One of the things investigated in this report is the required number of samples for Monte Carlo to give reliable results. This random number generation algorithm takes about twice as long to calculate than the standard random number generating algorithm, *ran0*, but *ran0* has a period of only 2×10^{9}. Unfortunately, the more random numbers that are needed, the longer it takes to calculate each random number.

The input variables for the clutch assembly are normally distributed, not uniformly distributed. The algorithm *gasdev* [Press 1992, 289] was used to map the uniform random deviate to the normal distribution. After the random inputs are ready for Monte Carlo, the next step is to use them in the assembly function.

Section 3.3. Explicit and Implicit Assembly Functions

Once the random numbers are generated and mapped onto the probability density functions for each of the independent variables, the values for the dependent variables and objective functions must then be calculated. As hundreds of thousands of samples may be required when using Monte Carlo, solving implicit equations may require significantly more computation time.

Glancy used Newton's method for solving systems of nonlinear assembly equations. He found that the clutch assembly problem required an average of 3.4 iterations to solve the implicit equations for each Monte Carlo simulation sample [Glancy 1994, 48]. Using explicit equations with the Monte Carlo method is extremely advantageous, but often not possible.

Section 3.4. Methods to Calculate Sensitivities

Sensitivity information is helpful when analyzing an assembly. The sensitivities reveal how the objective function changes as each of the independent variables change. For the clutch problem, the sensitivity of the contact angle with respect to the hub radius is the partial derivative $\partial\phi/\partial a$. The partial derivatives can be determined analytically or with a finite difference method.

The sensitivity information reveals the part features that might require more accurate processes. If the sensitivity is large, then variation of the input variable will be amplified to the assembly function. Percent contribution information can easily be estimated from the sensitivities and tells which variables with their variations do contribute the most to the overall assembly variance.

3.4.1. Sensitivities from the Correlation Coefficients

When using Monte Carlo simulation to analyze an assembly, sensitivity information can be obtained through using correlation coefficients and individual sampling. A linear correlation coefficient tell how strongly the output function changes as an input changes. The correlation coefficients range from –1.0 to 1.0. A value of 0.0 indicates that the output is completely uncorrelated to the input. A value of 1.0 indicates that there is an exact linear relationship between the output and the input variable, the output increasing as the input variable increases. A value of –1.0 also indicates an exact linear relationship, but with the output decreasing as the input variable increases.

One method of calculating the correlation coefficients is using the actual values of the input variables and the output to calculate the Pearson correlation coefficient (see Eq. 3-1). But sometimes the Pearson correlation coefficient can conclude that a correlation exists even if one does not because of the differences in magnitude between the input and the output [Press 1992, 640].

$$r_p = \frac{\sum_i (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_i (x_i - \bar{x})^2}\sqrt{\sum_i (y_i - \bar{y})^2}} \qquad \textbf{Eq. 3-1}$$

The Spearman rank-order correlation coefficient is a more robust method that uses the rank of the input variables and the rank of the output function (see Eq. 3-2). Even though the Spearman coefficients may be more robust (not measuring a correlation if none exists), there is a loss of information, as the actual x_i and y_i data is converted into R_i (the rank of the input variable) and S_i (the rank of the output variable). All of the values for the inputs and outputs must be stored during the simulation. Then the sets of values are all ranked according to magnitude. It is the rank of the variables used instead of the actual values.

$$r_s = \frac{\sum_i (R_i - \bar{R})(S_i - \bar{S})}{\sqrt{\sum_i (R_i - \bar{R})^2}\sqrt{\sum_i (S_i - \bar{S})^2}} \qquad \textbf{Eq. 3-2}$$

The program *Crystal Ball*, by Decisioneering Inc., uses the Spearman rank-order correlation coefficients to calculate the sensitivity information, and will be used to find the sensitivities for the clutch problem. In order to estimate the percent contribution, the rank correlation coefficients are squared and normalized to sum to 100%. This method is only an estimate, and not a true decomposition of variance as will be presented with Design of Experiments, in Chapter 6.

3.4.2. Sensitivities from Individual Sampling

Another method of calculating the sensitivities with the Monte Carlo is individual sampling. This method consists of holding all of the independent variables constant except for the one being studied. The value of the one variable is changed either to specific points (finite difference derivative) or to many random points (comparing the variance of the output to the variance of the input). Each of the other independent variables is analyzed similarly. A finite difference approach is very similar in that only one variable is changed at a time.

Section 3.5. The State-of-the-art in Estimating the Effectiveness of Monte Carlo

The error in using Monte Carlo to count the parts per million (PPM) rejects is found by way of the binomial distribution [Shapiro 1981, 292]. Expressing the error in PPM rejects as a percent of the rejects is shown below in Eq. 3-1. As the sample size (n) doubles, the error estimating the rejects decreases by about $\sqrt{2}$. *PPM* is an estimate of the parts per million rejects.

$$\left(\sigma_{PPM}\right)^2 = \frac{PPM\,(10^6 - PPM)}{n-1} \qquad \textbf{Eq. 3-1}$$

$$\sigma_{\%PPM} = \frac{\sigma_{PPM}}{PPM} = \sqrt{\frac{10^6 - PPM}{PPM\,(n-1)}}$$

Where: σ_{PPM} = The one-sigma bound on the error of rejects (where PPM in the equation is the initial estimate of the PPM rejects)

$\sigma_{\%PPM}$ = The one-sigma bound on percent error of PPM rejects

The one-sigma confidence limits calculated by Eq. 3-1 can be multiplied by any value to obtain the confidence level desired for the estimate of accuracy. The figures presented in this thesis will use the one-sigma confidence limits for the errors for easy comparison. To explain what the one-sigma error is, Figure 3-1 below shows a histogram of percent error for lower PPM rejects. Each of the calculations of percent error represents one cycle (or

run) of 10,000 Monte Carlo samples. A total of 1,000 runs were performed to generate the histogram. The assembly is the contact angle for the one-way clutch assembly, and the rejects correspond to assemblies with contact angles outside of the specification limits presented in Section 2.2. The PPM rejects correspond to about a 2.6 sigma quality level, or 4,600 PPM rejects.

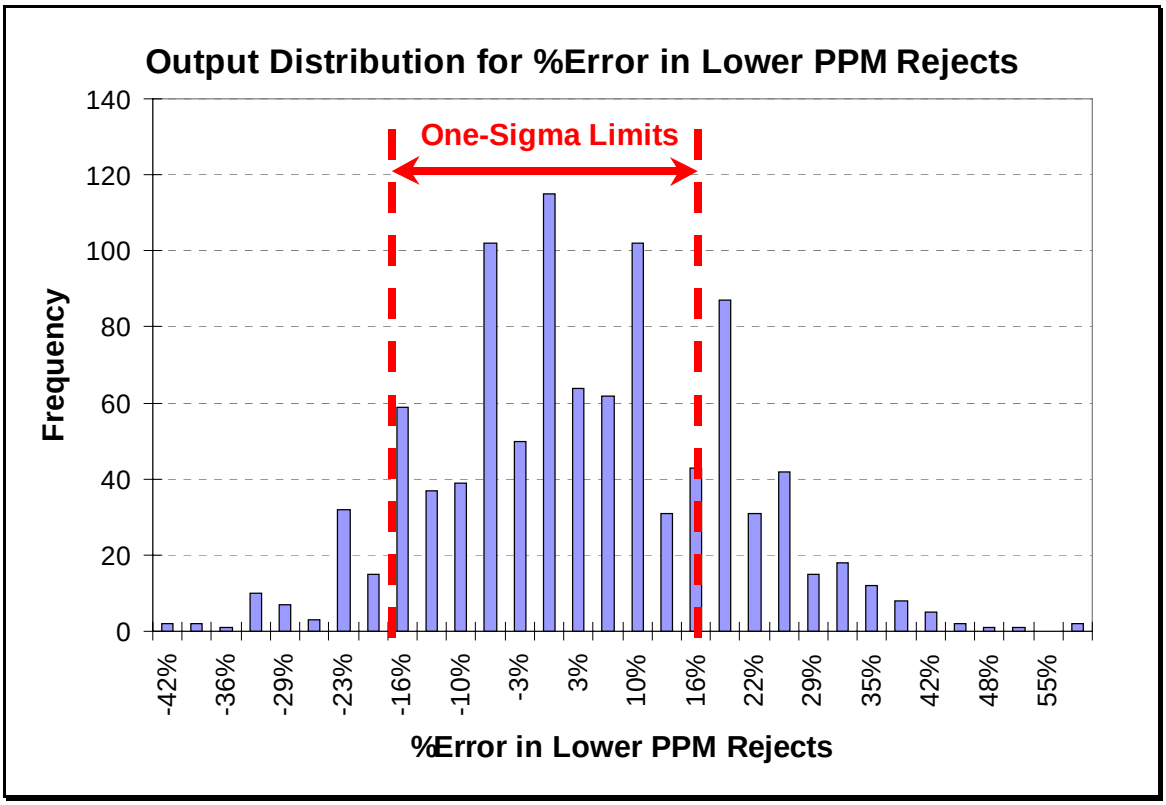


Figure 3-1: Histogram of %Error for Lower Rejects for 1,000 Different Runs (Simulations of Monte Carlo), each with 10,000 Samples

The one-sigma limits for the percent error were calculated with Eq. 3-1. The mean of the percent error for all 1,000 runs (or cycles of Monte Carlo) was about two percent. This histogram is portraying the one-sigma error limits, or bounds. The one-sigma limits are at about sixteen percent. If the sample size were doubled, then the one-sigma bound on the percent error ($\sigma_{\%PPM}$) would be divided by about the square root of two, and would then

be only eleven percent. Thus as the sample size increases, the histogram would become more narrow, and the accuracy increase.

Table 3-1 shows the statistics for the percent error of the lower, upper, and total PPM rejects for the 1,000 Monte Carlo simulations. The estimates of standard deviation of the percent errors are also compared with the estimates calculated based on Eq. 3-1.

Table 3-1: Distribution for %Error Counting Rejects for 10,000 Samples

	%Error in Lower PPM	%Error in Upper PPM	%Error in Total PPM
Mean (histogram)	2.2%	-9.9%	-2.1%
Stdev (histogram)	15.8%	20.0%	12.6%
$\sigma_{\%PPM}$ (equation)	**15.2%**	**20.4%**	**12.2%**

The standard deviation of the percent error for the histogram of 1,000 runs of 10,000 samples is very close to the estimate determined by Eq. 3-1. Even at 10,000 samples the one-sigma bound on the percent error for the lower PPM rejects is still about fifteen percent.

Section 3.6. Error in Estimating Moments with Monte Carlo

The estimate for mean and variance of the output distribution improves as the number of runs increases. The accuracy of the estimate of the mean of the output distribution is presented below in Eq. 3-1.

$$\frac{\hat{\mu}_1 - \mu_1}{\sigma/\sqrt{n}} = \text{Standard Normal Distribution} \qquad \textbf{Eq. 3-1}$$

$$\sigma_{\mu 1} = \frac{\sigma}{\sqrt{n}}$$

Where: $\hat{\mu}_i$ = The estimate of the i^{th} moment for a distribution

$\sigma_{\mu 1}$ = The one-sigma bound on the error of the estimation of the mean, also called the standard error of the mean [Creveling 1997, 63]

The standard error for the estimate of the mean decreases with the square root of *n* (sample size). The error for the estimate of the variance is a little more complicated. The distribution for estimating the accuracy of the variance is presented below in Eq. 3-2.

$$\frac{(n-1)\hat{\mu}_2}{\mu_2} = \text{Chi - Square Distribution :} \quad \text{mean} = (n-1),\ \text{variance} = 2(n-1) \qquad \textbf{Eq. 3-2}$$

The Chi-Square distribution for variance can be approximated by the normal distribution (with the same mean and variance) at large sample sizes [Vardeman 1994, 316], and Monte Carlo generally uses large sample sizes. Thus with the state-of-the-art, the accuracy of counting rejects, estimating the mean, and estimating the variance of a distribution can be evaluated. Chapter 9 will expand upon this to estimate the accuracy of estimating the skewness and kurtosis, as well as estimating the PPM rejects using the moments and the Lambda distribution.

Section 3.7. Advantages and Drawbacks of Monte Carlo Simulation

Monte Carlo is extremely versatile. Input variables may be correlated to each other, or in other words, not independent. Input variables may be of any distribution, even discrete. The number of rejects (assemblies outside of the specification limits) can be counted directly during the simulation, or the output distribution can be fitted to a standard distribution (normal) or a general distribution (Lambda for example) and the PPM rejects estimated for any limits.

One of the drawbacks is that large sample sizes are required, which may be time consuming if the assembly function is iterative. The smaller the percent rejects, the more samples are required for an accurate estimate. Once the simulation is performed, changes in the problem (changes in input variable distributions, changes in specification limits, etc.) most often require that the simulation be run again. The estimation of sensitivities is not a direct result of Monte Carlo. Although the variables could be changed one at a time or the correlation coefficients calculated to estimate the sensitivities.

Section 3.8. Sample Clutch problem

The one-way clutch assembly problem is easily modeled in a spreadsheet program (as the equation for the contact angle is explicit) and analyzed with *Crystal Ball*, a Monte Carlo simulation program from Decisioneering Inc. More difficult assemblies with iterative assembly functions would be more difficult to model, and would be impractical to use as an *Excel* model. Figure 3-1 below shows sample output of a Monte Carlo simulation on the clutch assembly. The histogram estimates the probability distribution for the contact angle ϕ.

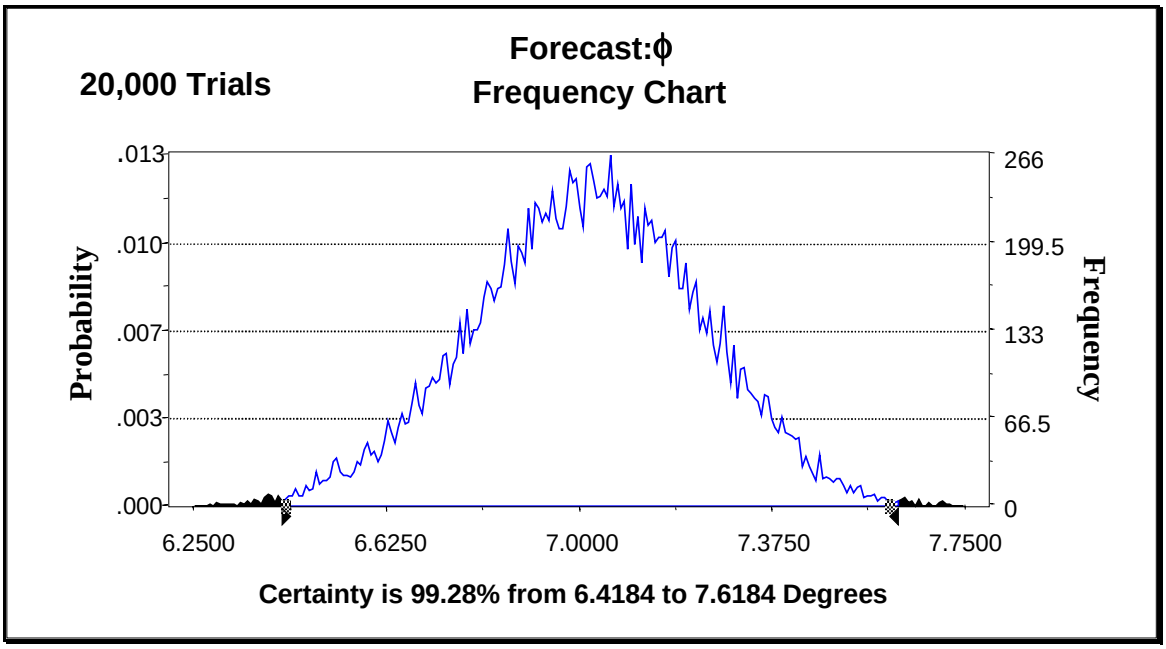


Figure 3-1: Monte Carlo Output from *Crystal Ball*

The histogram shows an estimation of the output distribution for the contact angle. With the specification limits as indicated, 99.28% of the sample assemblies are good, and 0.72% of the assemblies are outside the limits. The sample size was 20,000 for this run. The darkened areas of the tails of the histogram show the samples that fell outside the specification limits (rejects).

Using the Spearman correlation coefficients as described by Eq. 3-2, *Crystal Ball* estimates the sensitivities and the percent contribution of the independent variables as shown below in Table 3-1. This sensitivity information indicates that the variable **a** (hub radius) contributes most of the variation to the contact angle. A reduction in the variation of variable **a** should reduce the variation in the contact angle more than reductions in the variations of the other two variables.

Table 3-1: Sensitivity Information from *Crystal Ball*

Variable	Correlation Coefficient	%Contribution
a	-0.90	82.9%
c	-0.35	12.5%
e	0.21	4.5%

An estimation of the mean and variance and other moments of the distribution of the contact angle is also provided by *Crystal Ball*, but a custom Monte Carlo simulation program, using the random number generator described earlier, was used for flexibility in outputting the distribution moment information. Figure 3-2 shows how the estimates of the distribution moments change during the simulation as the sample size doubles. This simulation of over a billion samples, which took 8.6 hours to complete, will be used as the benchmark for accuracy of the other methods.

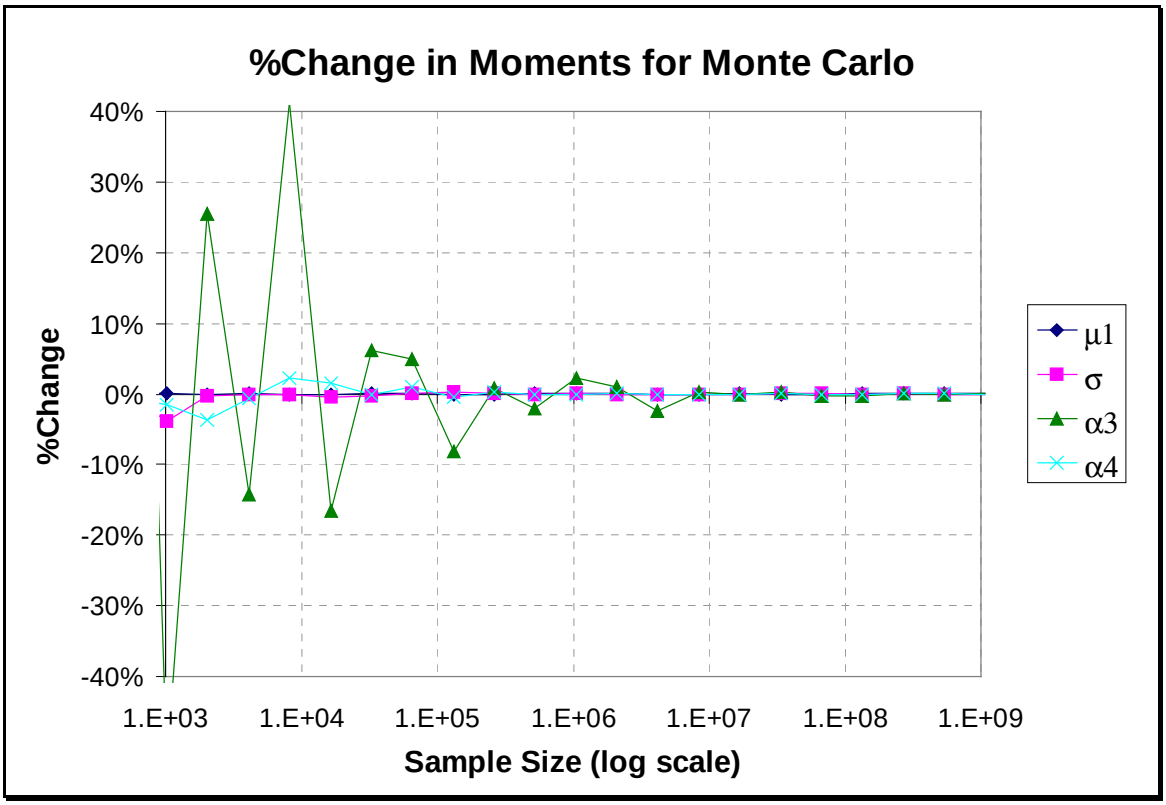


Figure 3-2: Percent Change in Moments for Monte Carlo Simulation as the Sample Size Doubles

The percent change for the mean is very little due to a large value of mean versus the standard deviation. The large change in skewness (α_3) is due to the small value of actual skewness. But it appears as though the error in estimating the skewness is greater than for kurtosis (this is not really the case though). A better method of calculating errors will be presented in Chapter 8. Using a percent change in the raw moments or the percent error in the moments is not a good estimate or accuracy.

As Figure 3-2 shows, the percent change in the moments as the sample size doubles is approximately zero as the sample size reaches a billion. The true values for the objective functions will be estimated by the values from this simulation. These moments, rejects, and quality loss for the clutch assembly are presented below in Table 3-2.

Table 3-2: Benchmark Results from Monte Carlo Simulation at One Billion Samples

Contact Angle for the Clutch	Value
$\mu 1$ (mean).....	7.014953
σ (Standard Deviation).....	0.219668
$\alpha 3$ (Skewness).....	-0.094419
$\alpha 4$ (Kurtosis).....	3.023816
Quality Loss ($/part).....	2.681
Lower Rejects (ppm).....	4,406
Upper Rejects (ppm).....	2,166
Total Rejects (ppm).....	6,572

The above values were obtained by using Monte Carlo Simulation at 2^{30} samples (over one billion). A description of how to calculate the quality loss is given in Chapter 6 as it is mainly used with the design of experiments.

Section 3.9. Confidence Interval for PPM Rejects Estimated

The $\sigma_{\%PPM}$ is easily calculated from Eq. 3-1 to be 0.0004, or 0.04 percent (using the estimate of total PPM rejects in the table above). That means that the 68 percent confidence level (plus and minus one σ_{PPM}) for the total PPM rejects is from 6,569 to 6,574. Similarly, if greater confidence is desired in the total PPM rejects, the 99.7 percent confidence interval (plus and minus three σ_{PPM}) for the rejects is 6,564 to 6,579 rejects. At one billion samples, the results are quite confident. The estimation of the confidence interval for the mean and variance is just as easily calculated from Eq. 3-1 and Eq. 3-2.

Monte Carlo Simulation is a very effective way to simulate the randomness that occurs in nature. The required sample size for a Monte Carlo simulation depends on the desired output of the analysis (mean, variance, quality loss, rejects, etc.) and the desired accuracy of that estimate. More on the required sample size for the Monte Carlo simulation method is presented in Chapter 9.

Section 3.10. Summary for Monte Carlo Simulation

The research and experimentation indicates that Monte Carlo simulation is the most flexible of the methods in terms of different types of input, output, and assembly functions. Input variables can be skewed or even correlated. The assembly function can even be discontinuous. Simulation is definitely the easiest method to understand. The major weaknesses of simulation are the number of trials necessary for accuracy, iterative assembly functions, and calculating the sensitivities of the inputs [Creveling 1997; Shapiro 1981].

Monte Carlo simulation is a valuable tool in variation analysis and is widely used. It has many advantages as well as limitations. Monte Carlo is a simulation method that attempts to model the randomness that occurs in nature. Because of its use of random numbers, more state-of-the-art techniques for estimating its accuracy are available than any other method, as will be shown in later chapters.

This chapter has not only provided the state-of-the-art on Monte Carlo and its accuracy, but it also has demonstrated its use on the sample clutch problem to allow for direct comparison with the other methods.

Chapter 4. Method of System Moments

This chapter introduces and demonstrates the method of system moments (MSM). Because MSM is not a very common method the background will also be discussed. The method of system moments is an advanced variation analysis method that uses the first and second derivatives of the assembly function with respect to the input variables, along with the first eight moments of the input variable probability distributions, to estimate the moments of the output distribution.

The Taylor series is used to estimate how the derivatives and input moments will affect the resultant assembly distribution. Often, only the first order terms of the Taylor series are used, but the second order terms can also be used to increase the accuracy of the output distribution prediction [Cox 1986; Shapiro 1981].

Once the output distribution moments are estimated, a Lambda distribution (discussed in Chapter 7), a Pearson distribution [Freimer 1988, 3560], or another distribution may be used to estimate percent rejects, quality loss, or other objective functions.

Section 4.1. Background of Method of System Moments

The method of system moments (MSM) estimates the assembly function with a multivariate Taylor series approximation. Eq. 4-1 below shows the general form of a second order Taylor expansion. For linear variation analysis, the second order terms are assumed to be negligible and thus eliminated [Cox 1986, 4–5].

$$R = R_0 + \sum_{i=1}^{n}\left(\frac{\partial h}{\partial x_i}\right)\left(x_i - x_{i0}\right) + \sum_{i=1}^{n}\frac{1}{2}\left(\frac{\partial^2 h}{\partial x_i^2}\right)\left(x_i - x_{i0}\right)^2 + \sum_{i=1}^{n-1}\sum_{j>i}^{n}\left(\frac{\partial^2 h}{\partial x_i \partial x_j}\right)\left(x_i - x_{i0}\right)\left(x_j - x_{j0}\right) + \textit{third order and higher terms} \qquad \textbf{Eq. 4-1}$$

Where: $R = h(x_i)$ = Value of the assembly function value evaluated at the x_i

$R_0 = h(x_0)$ = Value of the function evaluated at the mean values of the inputs, x_0

h = The assembly function

x_i = The value of the i^{th} input variable

x_{i0} = The i^{th} input variable at its mean value

In order to simplify the expressions for the finding the moments of the assembly function the following substitutions are made:

$$X_i = \frac{(x_i - x_{i0})}{\sigma_i} \qquad b_{ii} = \frac{1}{2}\left(\frac{\partial^2 h}{\partial X_i^2}\right)\sigma_i^2$$

$$b_i = \left(\frac{\partial h}{\partial X_i}\right)\sigma_i \qquad b_{ij} = \left(\frac{\partial^2 h}{\partial X_i \partial X_j}\right)\sigma_i\sigma_j$$

Eq. 4-2

The reason for these substitutions is to transform the distributions to non-dimensional coordinates, where the mean is 0.0 and the variance is 1.0. Truncating the terms of third order and higher and making the above substitutions, Eq. 4-1 simplifies into Eq. 4-3.

$$R \approx R_0 + \sum_{i=1}^{n}\left(b_i x_i + b_{ii} x_i^2\right) + \sum_{i=1}^{n-1}\sum_{j>i}^{n} b_{ij} x_i x_j$$

Eq. 4-3

The expected values of the moments of Eq. 4-3 are then the estimates for the output distribution. Generally, either the quadratic terms are used for a second-order analysis, or the just the linear terms are used for a first-order analysis.

Section 4.2. Second-order MSM Analysis

The resulting equations for calculating the mean and the variance of a quadratic assembly function are found below in Eq. 4-1. The equations for calculating the skewness and kurtosis of a quadratic equation are quite complex, and although they were coded in the analysis program used for this thesis, they will not be presented here. Refer to the book by Cox [Cox 1986, 49–50] or by Shapiro [Shapiro 1981, 357] for the full quadratic variation analysis expressions for skewness and kurtosis.

Eq. 4-1

$$\mu_1 = R_0 + \sum_{i=1}^{n} b_{ii}$$

$$\mu_2 = \sum_{i=1}^{n} \left[b_i^2 + 2 b_i b_{ii} (\alpha_3)_i + b_{ii}^2 (\alpha_4)_i \right] + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left[2 b_{ii} b_{jj} + b_{ij}^2 \right] - (\mu_1 - R_0)^2$$

The method of system moments provides a method to convert sensitivities and input variable moment information into moments for the output distribution. The difficulty lies in calculating the sensitivities.

Section 4.3. Linear Variation Analysis

For linear variation analysis, only the first order terms are retained in Eq. 4-3, and if the input variables are assumed to be independent (uncorrelated with each other), the following equations represent the first four moments of the output function [Cox 1986, 10].

Eq. 4-1

$$\mu_1 = R_0 \qquad \mu_3 = \sum_{i=1}^{n} b_i^3 (\alpha_3)_i$$

$$\mu_2 = \sum_{i=1}^{n} b_i^2 \qquad \mu_4 = \sum_{i=1}^{n} b_i^4 (\alpha_4)_i + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} 6 b_i^2 b_j^2$$

Determining when to use the linear and when to use the quadratic MSM estimate is difficult. The state-of-the-art is to try them both and see if the linear is a good enough approximation.

Section 4.4. Calculating the Sensitivities for the Clutch Problem

Both Eq. 4-1 for second-order analysis and Eq. 4-1 for linear analysis include terms for the partial derivatives of the desired assembly function with respect to the input variables at their mean values. These sensitivities (first and second derivatives) could be found by differentiating the explicit equation for the output function if it were simple enough.

Alternately, for implicit equations and more complex explicit assembly functions, a finite difference method may be used to determine the derivatives numerically. Because the clutch assembly can be represented by an explicit equation for the contact angle, Eq. 2-1 will be differentiated symbolically and then evaluated at the mean values of the input variables. Eq. 4-1 below shows the analytical solutions to the first derivatives.

Eq. 4-1

$$\frac{\partial \phi}{\partial a} = \frac{-1}{\sqrt{(e-c)^2-(a+c)^2}} = -0.207877\,\frac{rad}{mm} = -11.9104\,\frac{\deg}{mm}$$

$$\frac{\partial \phi}{\partial c} = \frac{-1}{\sqrt{(e-c)^2-(a+c)^2}}\left(1+\frac{a+c}{e-c}\right) = -23.7317\,\frac{\deg}{mm}$$

$$\frac{\partial \phi}{\partial e} = \frac{1}{\sqrt{(e-c)^2-(a+c)^2}}\left(\frac{a+c}{e-c}\right) = 11.8212\,\frac{\deg}{mm}$$

From an examination of the magnitudes of the first derivatives calculated above, the contact angle is most sensitive to changes in the roller radius ***c***. The standard deviation of the hub radius is about five times that of the roller radius, thus accounting for the larger percent contribution of the hub radius. The second derivative sensitivities are shown below in Eq. 4-2.

Eq. 4-2

$$\frac{\partial^2 \phi}{\partial a^2} = -(a+c)\left[(e-c)^2-(a+c)^2\right]^{-\frac{3}{2}} = -20.11133\,\frac{\deg}{mm^2}$$

$$\frac{\partial^2 \phi}{\partial c^2} = -(a+e)\left(1+\frac{a+c}{e-c}\right)\left[(e-c)^2-(a+c)^2\right]^{-\frac{3}{2}}$$

$$-\left[(e-c)^2-(a+c)^2\right]^{-\frac{1}{2}}\left[\frac{1}{e-c}+\frac{a+c}{(e-c)^2}\right] = -81.04923\,\frac{\deg}{mm^2}$$

$$\frac{\partial^2 \phi}{\partial e^2} = -(a+c)\left[(e-c)^2-(a+c)^2\right]^{-\frac{3}{2}}$$

$$-\left[(e-c)^2-(a+c)^2\right]^{-\frac{1}{2}}\frac{(a+c)}{(e-c)^2} = -20.41158\,\frac{\deg}{mm^2}$$

The method of system moments not only requires the first and second derivatives for the full quadratic model, but it also requires the cross-derivatives. Eq. 4-3 below shows the calculations of the three cross-derivatives for the contact angle.

$$\frac{\partial^2\phi}{\partial a\partial c} = -(a+e)\left[(e-c)^2-(a+c)^2\right]^{-\frac{3}{2}} \qquad = -40.37448\frac{\deg}{mm^2}$$

$$\frac{\partial^2\phi}{\partial a\partial e} = (e-c)\left[(e-c)^2-(a+c)^2\right]^{-\frac{3}{2}} \qquad = 20.26316\frac{\deg}{mm^2}$$

$$\frac{\partial^2\phi}{\partial c\partial e} = (e-c)\left(1+\frac{a+c}{e-c}\right)\left[(e-c)^2-(a+c)^2\right]^{-\frac{3}{2}}$$

$$+\left[(e-c)^2-(a+c)^2\right]^{-\frac{1}{2}}\frac{(a+c)}{(e-c)^2} = 40.67474\frac{\deg}{mm^2}$$

Eq. 4-3

Finding the analytical derivatives of Eq. 2-1 to get the second order sensitivities is not easy, even though the equation for the contact angle was relatively simple. For more complex assemblies, a finite difference method would be preferred. Chapter 9 and Chapter 12 will present numerical methods for estimating the sensitivities of assembly functions that provide even more accurate results when used with MSM.

Section 4.5. Sample Clutch Problem Using Method of System Moments

In order to perform the method of system moments (MSM) on the clutch problem, the sensitivities calculated in the previous section were used, and the input distributions were assumed to be normally distributed with the standard deviations as presented earlier in Table 2-2. The method of system moments for second order variation analysis requires that the first eight moments of the input distributions be known. The first eight moments of a standard normal distribution are 0, 1, 0, 3, 0, 15, 0, 105, respectively. In many cases, only the first few moments of the input distributions may be known with any level of certainty. Chapter 9 will present a method of estimating the higher moments of a distribution by fitting a Lambda distribution to the known moments.

Table 4-1: Analysis Results from the Second Order Method of System Moments

Contact Angle for the Clutch	Monte Carlo	MSM 2nd Order	%Error
$\mu 1$ (mean).....	7.014953	7.014969	0.0002%
σ (Standard Deviation).....	0.219668	0.219345	-0.147%
$\alpha 3$ (Skewness).....	-0.094419	-0.093555	-0.915%
$\alpha 4$ (Kurtosis).....	3.023816	3.011671	-0.402%
Quality Loss ($/part).....	2.681	2.674	-0.294%
Lower Rejects (ppm).....	4,406	4,180	-5.13%
Upper Rejects (ppm).....	2,166	2,322	7.21%
Total Rejects (ppm).....	6,572	6,502	-1.06%

The estimate of the mean for the contact angle probability distribution is very accurate. The estimates of the other moments are not as good. The true value of skewness (α_3) is so small that the error in skewness is amplified when calculating percent error. The estimates for rejects were obtained by fitting a Lambda distribution to the moments (method described in Chapter 7) and solving for the rejects. The error for total rejects is significantly less than the error for either the upper of lower rejects alone.

The second order method of system moments estimates the first four moments for the clutch contact angle within one percent of the true moments (estimated by Monte Carlo at a billion samples). The second order method approximates the assembly function as a quadratic function, so if the assembly function were quadratic, the results would be as accurate as the accuracy of the input data and the round-off error permit. The first order method of system moments will be discussed in Chapter 5, along with a method of transforming a set of implicit equations into an explicit linear approximation.

Section 4.6. State of Art for Estimating the Accuracy of MSM

There is no method available to estimate the accuracy of the MSM without comparing the results to the Monte Carlo benchmark results. The accuracy depends on how non-quadratic the assembly function is. About the only thing that is known without performing a full Monte Carlo simulation is that second order MSM is more accurate than

first order analysis. The hybrid method that uses MSM presented in Chapter 12 will have an estimate for accuracy that the regular MSM does not.

Section 4.7. Summary for the Method of System Moments

The major strengths of the method of system moments are clear and accurate estimates of input sensitivities and flexibility of the input and output distributions. The weaknesses of the method of system moments are the difficulty in calculating the derivatives of complex assembly functions, extreme complexity of calculating the higher moments for the second-order Taylor approximation, and the difficulty of estimating the accuracy of the analysis.

The method of system moments is not commonly used, and therefore the method itself is state-of-the-art. But MSM is valuable because it is very different than the other methods. It does not require the assembly function but the derivatives. If the second-order derivatives are not available, the analysis can still be performed assuming linearity. Because of this uniqueness it is very easily incorporated into a hybrid method. Section 10.1 will combine MSM with a response surface method, and Chapter 12 will combine them both with Monte Carlo.

Chapter 5. RSS First-order Approximation

This chapter will explain the method of linear variation analysis, or RSS (root-sum-square). The purpose of this chapter is to demonstrate how assuming linearity simplifies the analysis to the point that it is quite easy to perform and understand. The state-of-the-art in terms of RSS is the method to obtain the sensitivities from the implicit equations. This method will be demonstrated. One of the main contributions of this thesis is to present a method of estimating the accuracy of linear approximation (presented in Section 10.2).

Section 5.1. Background of RSS First-order Approximation

The RSS method is based on the Method of System Moments using only the first order terms of the Taylor series. The general equations for the linear MSM were shown by Eq. 4-3. But for the RSS method, the output distribution is generally assumed to be normal. Thus the only thing to calculate is the variance of the output distribution because the mean of the linear analysis is the nominal.

Calculation of the variance of the output distribution is based on the classical statistical theorem that variances add linearly.

$$o^2_{Assembly} = o^2_{Parts} \qquad \textbf{Eq. 5-1}$$

$$o^2_{Assembly} = \sum_{i=1}^{n} \left(S_i \sigma_i\right)^2$$

Where: S_i = The first-order sensitivity (derivative) of the function with respect to the i^{th} input variable

σ_i = The standard deviation of the i^{th} input variable

RSS can be used with any function, even if it is not linear. Section 10.2 will discuss how to estimate the error of assuming a non-linear assembly is linear. The justifications for assuming the output distribution is normal are listed below.

- If the input distributions are normal and the assembly function linear, the output distribution will be normally distributed.

- If the assembly function is linear, then even non-normal distributions for the components add to a normal distribution as the number of components get large (central limit theorem). Even as few components as five often approaches normality [Chase 1991, 29].

- Information on the skewness and kurtosis of input variables is not generally available. And when there is different from a normal distribution, it is often not statistically significant. Statistical process control data generally only gathers information on the mean and variance.

Estimates of the input sensitivities can either be obtained from the explicit assembly functions, or from implicit equations calculated directly from the assembly geometry. These estimates can then be used to calculate the variation of the assembly characteristics and other output functions of interest [Chase 1995].

Section 5.2. First-order versus Second-order

Second order variation analysis models many assembly problems better than first order. But one advantage of first-order variation analysis is that a set of implicit equations that describe an assembly function can be expressed as a set of linear equations that can be solved explicitly through linear algebra. The second-order sensitivities are not required for RSS.

Section 5.3. Linear Explicitization

Implicit equations that describe an assembly function are written in the general form expressed below in Eq. 5-1. The variables x_i represent the n independent variables. The

variables u_j represent the m dependent variables. And the equations h_k represent the l implicit equations, where l is greater than or equal to m.

$$\begin{aligned} h_1(x,u) &= h_1(x_1, x_2, \cdots, x_n, u_1, u_2, \cdots, u_m) \\ h_2(x,u) &= h_2(x_1, x_2, \cdots, x_n, u_1, u_2, \cdots, u_m) \\ &\vdots \\ h_l(x,u) &= h_l(x_1, x_2, \cdots, x_n, u_1, u_2, \cdots, u_m) \end{aligned} \qquad \textbf{Eq. 5-1}$$

Where: The assembly equations equal zero for closed loop constraint equations, and equal the value of the nominal plus and minus the variation for the open loop equations.

A first order Taylor series expansion (as described by Eq. 4-1) will be performed on the implicit equations above to linearize them. This approximation is good for small perturbations around the nominal values of the input variables. Eq. 5-2 below shows this expansion.

$$h_k = h_{k0} + \sum_{i=1}^{n} \left(\frac{\partial h_k}{\partial x_i} \right) (x_i - x_{i0}) + \sum_{j=1}^{m} \left(\frac{\partial h_k}{\partial u_j} \right) (u_j - u_{j0}) \qquad \textbf{Eq. 5-2}$$

Where: h_{k0} = The nominal value of the kth assembly function h_k (zero if closed loop constraint)

x_{i0} = The nominal value of the ith independent variable x_i

u_{j0} = The nominal value of the jth dependent variable u_j

Once the linear Taylor expansion is performed, the set of implicit equations can be expressed as a linear algebraic equation in matrix form, as shown in Eq. 5-3 below.

$$\begin{aligned} 0 &= A\Delta x + B\Delta u \\ &\textit{or} \\ \Delta u &= -\left(B^{-1}A\right)\Delta x = S\Delta x \end{aligned} \qquad \textbf{Eq. 5-3}$$

Where: Δx = A vector of the independent variables ($x_i - x_{i0}$). It can also be the vector of variations for the independent variables as the equation is linear.

Δu = A vector of the dependent variables ($u_j - u_{j0}$). It can also be the vector of the resultant variations for the dependent variables, because the equation is linear.

$$A = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \cdots & \frac{\partial h_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial h_l}{\partial x_1} & \frac{\partial h_l}{\partial x_2} & \cdots & \frac{\partial h_l}{\partial x_n} \end{bmatrix} \quad B = \begin{bmatrix} \frac{\partial h_1}{\partial u_1} & \frac{\partial h_1}{\partial u_2} & \cdots & \frac{\partial h_1}{\partial u_m} \\ \frac{\partial h_2}{\partial u_1} & \frac{\partial h_2}{\partial u_2} & \cdots & \frac{\partial h_2}{\partial u_m} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial h_l}{\partial u_1} & \frac{\partial h_l}{\partial u_2} & \cdots & \frac{\partial h_l}{\partial u_m} \end{bmatrix}$$

$$S = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial u_m}{\partial x_1} & \frac{\partial u_m}{\partial x_2} & \cdots & \frac{\partial u_m}{\partial x_n} \end{bmatrix}$$

Note: the *S* matrix is easily obtained from the *A* and *B* matrices and contains the sensitivities for linear variation analysis.

The *A* and *B* matrices are not the sensitivities of the dependent variables to the independent variables, but intermediate matrices. The $-(B^{-1}A)$ or *S* matrix is easily obtained through matrix manipulation methods, and is the sensitivity matrix for linear variation analysis.

Section 5.4. First Order Variation Analysis

The *S* matrix contains the partial derivatives of each of the dependent variables with respect to each of the input variables. Eq. 5-1 below shows how the *S* matrix can be used to calculate the variations of the dependent variables.

$$\sigma(u_j) = \sqrt{\sum_{i=1}^{n} \left(S_{ji}\sigma(x_i)\right)^2} \qquad \textbf{Eq. 5-1}$$

For a linear variation analysis, the mean of the assembly function is assumed to be the value of the assembly function with the independent variables at their nominal (or mean) values. Additionally, the output function is assumed to be normally distributed. David Larsen [Larsen 1989, 61] found that the greater the number of independent variables, the nearer to normal the output function would be. If all of the input variables are in fact

normally distributed, then Eq. 4-1 for linear method of system moments predicts that the output distribution will be normally distributed as well.

Section 5.5. Sample Clutch Problem using Linear Explicitization and RSS

The first step in performing a linear variation analysis is to find the implicit equations for the assembly function. Section 2.4 showed the creation of the vector loop and resulting set of implicit equations. The loop is a closed loop so each of the assembly equations is equal to zero. These equations will be represented here as Eq. 5-1 for a quick reference.

$$\begin{aligned} h_x &= 0 = b + c\sin(\phi_1) - e\sin(\phi_1) \\ h_y &= 0 = a + c + c\cos(\phi_1) - e\cos(\phi_1) \\ h_\theta &= 0 = -\phi_1 + \phi_2 \end{aligned} \qquad \textbf{Eq. 5-1}$$

Determining the partial derivatives from these implicit equations is much easier than from the explicit equations (refer to Eq. 4-1). The *A* and *B* matrices are obtained by taking the partial derivatives of the assembly equations above. The *S* matrix is then obtained by numerically computing $-(B^{-1}A)$. Once the partial derivatives are taken, the *A* and *B* matrices are the following (refer to Eq. 5-3 for explanations).

$$A = \begin{bmatrix} \frac{\partial h_x}{\partial a} = 0 & \frac{\partial h_x}{\partial c} = \sin(\phi_1) & \frac{\partial h_x}{\partial e} = -\sin(\phi_1) \\ \frac{\partial h_y}{\partial a} = 1 & \frac{\partial h_y}{\partial c} = 1 + \cos(\phi_1) & \frac{\partial h_y}{\partial e} = -\cos(\phi_1) \\ \frac{\partial h_\theta}{\partial a} = 0 & \frac{\partial h_\theta}{\partial c} = 0 & \frac{\partial h_\theta}{\partial e} = 0 \end{bmatrix} = \begin{bmatrix} 0 & .1222 & -.1222 \\ 1 & 1.992 & -.9925 \\ 0 & 0 & 0 \end{bmatrix} \qquad \textbf{Eq. 5-2}$$

$$B = \begin{bmatrix} \frac{\partial h_x}{\partial b} = 1 & \frac{\partial h_x}{\partial \phi_1} = (c-e)\cos(\phi_1) & \frac{\partial h_x}{\partial \phi_2} = 0 \\ \frac{\partial h_y}{\partial b} = 0 & \frac{\partial h_y}{\partial \phi_1} = -(c-e)\sin(\phi_1) & \frac{\partial h_y}{\partial \phi_2} = 0 \\ \frac{\partial h_\theta}{\partial b} = 0 & \frac{\partial h_\theta}{\partial \phi_1} = -1 & \frac{\partial h_\theta}{\partial \phi_2} = 1 \end{bmatrix} = \begin{bmatrix} 1 & -39.37 & 0 \\ 0 & 4.810 & 0 \\ 0 & -1 & 1 \end{bmatrix} \qquad \textbf{Eq. 5-3}$$

Using the above matrices to calculate the *S* matrix as $-(B^{-1}A)$, the resulting sensitivity matrix is shown below as Eq. 5-4.

$$S = -\left(B^{-1}A\right) = \begin{bmatrix} \frac{\partial b}{\partial a} = -8.12279 & \frac{\partial b}{\partial c} = -16.30691 & \frac{\partial b}{\partial e} = 8.18412 \\ \frac{\partial \phi_1}{\partial a} = -0.20788 & \frac{\partial \phi_1}{\partial c} = -0.41420 & \frac{\partial \phi_1}{\partial e} = 0.20632 \\ \frac{\partial \phi_2}{\partial a} = -0.20788 & \frac{\partial \phi_2}{\partial c} = -0.41420 & \frac{\partial \phi_2}{\partial e} = 0.20632 \end{bmatrix} \quad \textbf{Eq. 5-4}$$

The linearization and sensitivity calculations are straightforward and easily automated. Additionally, feature control variations due to shape, orientation, and position variations may also be inserted into the vector diagrams.

For linear RSS variation analysis, the calculation of the variance of the contact angle for the clutch is shown in Table 5-1 below. Each of the sensitivities is multiplied by the standard deviation for each of the independent variables, and then squared to yield the contribution to variance. The sum of the contribution to variance for each of the independent variables is the variance of the contact angle. The ratio of the contribution to variance to the total variance is the percent contribution.

Table 5-1: Calculation of the Variance of the Contact Angle

	Sensitivity (rad/mm)	Sensitivity (deg/mm)	σ(degrees)	Contribution to Variance	Percent Contribution
a	-0.20788	-11.91065	0.01666667	0.039406529	**81.94%**
c	-0.41420	-23.73191	0.00333333	0.006257818	**13.01%**
e	0.20632	11.82127	0.00416667	0.002426082	**5.04%**
Total				**0.048090429**	100%

The hub radius ***a*** contributes over 80 percent of the total variation of the contact angle. The sensitivity of the hub radius is not the largest, but the input variation of ***a*** is the largest. The percent contributions calculated by the linear RSS method and Monte Carlo are very similar. Table 5-2 below shows that the difference is less than one percent. The sensitivity calculations by Monte Carlo can not be assumed to be the correct values

because they were calculated using a rank correlation coefficient. While the rank correlation coefficient are very robust, they are not necessarily accurate enough to be the benchmark sensitivity estimates.

Table 5-2: Percent Contribution from RSS Linear and Monte Carlo

	Percent Contribution		
	RSS (linear)	Monte Carlo	Difference
a	81.94%	82.94%	-1.00%
c	13.01%	12.54%	0.47%
e	5.04%	4.52%	0.53%

Table 5-3 below compares the results (output distribution moments, quality loss, and PPM rejects) from the linear RSS method with the benchmark results from Monte Carlo.

Table 5-3: Analysis Results from RSS Linear

Contact Angle for the Clutch	Monte Carlo	RSS Linear	%Error
$\mu 1$ (mean).....	7.014953	7.018400	0.049%
σ (Standard Deviation).....	0.219668	0.219295	-0.170%
$\alpha 3$ (Skewness).....	-0.094419	0.000000	-100%
$\alpha 4$ (Kurtosis).....	3.023816	3.000000	-0.788%
Quality Loss ($/part).....	2.681	2.672	-0.364%
Lower Rejects (ppm).....	4,406	3,109	-29.4%
Upper Rejects (ppm).....	2,166	3,109	43.5%
Total Rejects (ppm).....	6,572	6,219	-5.38%

The RSS linear approximation gives a good estimate for the mean, standard deviation, quality loss, and the total rejects. The rejects were calculated assuming the output was normally distributed. Even with a highly non-linear problem, and only three independent variables, the kurtosis is very similar to that of a normal distribution. Additionally, the estimate of total rejects using the normal distribution is close to the actual. The lower or upper rejects alone is not well approximated using the RSS linear and the normal distribution because of both the shift in the mean from the nominal and the skewness of the output distribution.

Linear variation analysis does not perform too badly on this extremely non-linear problem. Assuming that the distribution had no skewness does not affect the total rejects too much because the error in upper rejects partially offsets the error in lower rejects. It seems to generally be the case that the error in total rejects is less than the error in one of the tails (if the specification cut-off points are centered around the mean).

Section 5.6. Summary of RSS Method

Unlike Monte Carlo and second order method of system moments, linear variation analysis does not require second order sensitivities or detailed knowledge of the input distributions. The accuracy of the linear variation analysis is definitely less than the other two (depending on the sample size for Monte Carlo), but for many situations the accuracy would be more than adequate.

The major strengths of the RSS methods are the ease of automation and the ability to linearize extremely complicated assembly functions, while incorporating kinematic assembly variation into the model. For this reason, the RSS method is most useful with implicit functions, eliminating the need for iterative solutions. The major weaknesses of the RSS model are incorporating correlation of input variables, accounting accurately for non-normal input variables, and estimating the error of the analysis.

Accuracy out the third or fourth decimal is not always required. Section 10.2.1 discusses a method of estimating the error of linearization to help determine if the accuracy is good enough. Additionally, Chapter 13 presents a framework for estimating the inherent error in the problem due to the accuracy of the input information. Due to these contributions, the linear RSS method may be the preferred method in many variation analysis situations.

Chapter 6. Quality Loss and Design of Experiments

This chapter will investigate the methods of using design of experiments for assembly variation analysis. Its use in this area has only recently become popular. Not much state-of-the-art information exists on estimating the accuracy of an experimental design without performing the design and comparing it to a more complete design or Monte Carlo. Section 10.2 will contribute a method to help optimize the design for accuracy while limiting the number of runs.

Design of experiments is a generalized method for examining the variability in any process. It consists of measuring the output function for specific combinations of input variable values. When applied to assembly variation analysis, the procedure may be used to analyze variability in an analytical assembly function, in which calculated responses to variation replace experimentally measured outputs [Creveling 1997; D'Errico 1988; Gerth 1997; Mori 1990].

The set of output trials can then be analyzed to estimate the overall output distribution and the effects of each of the input variables and the interactions. These effects can also be used to estimate the partial derivatives of the assembly function. Orthogonal arrays with reduced numbers of trials can be used to improve the efficiency of the process [Taguchi 1986; Taguchi 1993]. A response surface can also be fit through the points to facilitate additional understanding of the function [Myers 1971; Myers 1995].

The quality loss function in state-of-the-art literature has only really been used with design of experiments. It is an excellent tool, and one value of this thesis is showing that it can effectively be used with any of the variation analysis methods.

Section 6.1. The Taguchi Quality Loss Function

The normal paradigm for product acceptance is to think of products as good if they are within specification and bad if they are not. This kind of thinking results in a discontinuous system in which a part barely inside the specification limits is good, while a part barely outside the limits is totally bad. Of course the accuracy in measuring the assemblies would allow for some parts outside the limits to be accepted and vice-versa. To estimate the quality loss from assemblies that deviate from optimal, Taguchi uses a quadratic loss function based on cost. Figure 6-1 is a visual comparison between the quadratic loss function and the step function (percent rejects).

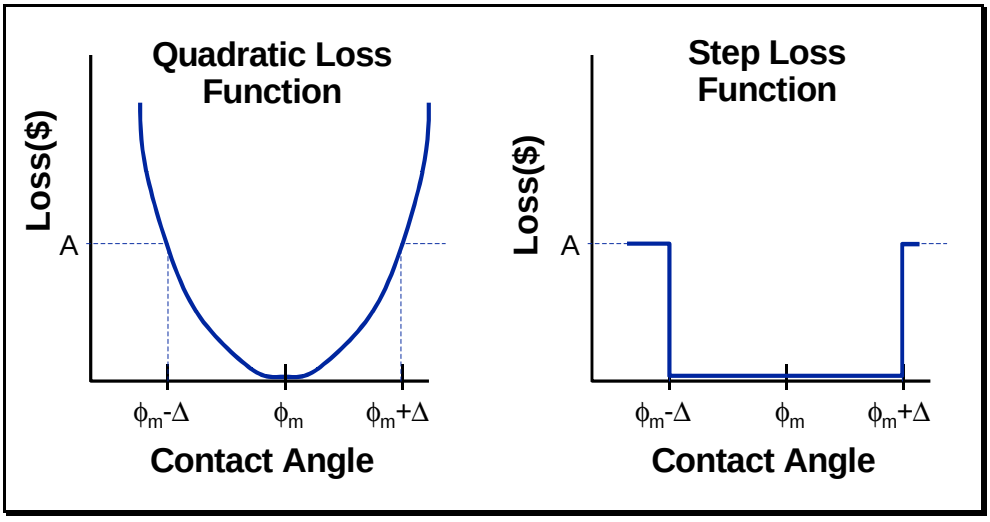


Figure 6-1: Comparison between Quadratic and Step Loss Functions

The quadratic loss function has a point where the quality loss is a minimum. At every other point the loss is greater, increasing farther from the optimum. Specification limits result in a step loss function which assumes that between the specification limits there is no quality loss and anywhere outside the specification limits is equally bad.

In the case of the clutch assembly, if the contact angle is 7.0784 degrees exactly (upper specification limit), is it good or bad? Suppose that the specification limits were chosen at points where the clutch would not function properly under a maximum operating load

plus a safety factor. If the clutch performs best at the optimum contact angle of 7.0184 degrees (ϕ_m), the performance would thus deteriorate with any deviation from the optimum contact angle. Thus an assembly with a contact angle of 7.0784 is relatively bad.

Variation from the optimum might not cause "rejects," but it might lower the average performance, and thus lower the market price for the clutches. Lower average quality might require more inspection of the clutches, increasing costs. Abnormal operating conditions for the clutches might cause "good" clutches to fail. Sub-optimal clutches might require more maintenance, thus lowering the value of the clutches to the customer. All of these extra losses due to quality cause the quality loss to increase within the specification limits. So the quadratic loss function is more realistic than the step function.

Determining the complete "cost to society," as Taguchi advocates, seems impossible, but an accurate cost estimate is not as important as having a relative cost function by which to compare options. The general loss function is described by Eq. 6-1 below as if the assembly was the one-way clutch. The integration is over all of the possible output contact angles.

$$L = K \int (\phi - \phi_m)^2 f(\phi) d\phi \qquad \textbf{Eq. 6-1}$$

Where: L = The quality loss

K = The cost constant

ϕ = *The contact angle for an assembly*

ϕ_m = *The optimal contact angle for minimum loss*

$f(\phi)$ = *The probability density function for the contact angle (output distribution)*

Figure 6-2 below shows some of the terms on a graphical representation of both the loss function and a probability distribution. Integrating the product of the two determines the average quality loss per part (the integral of the probability density function is 1.0).

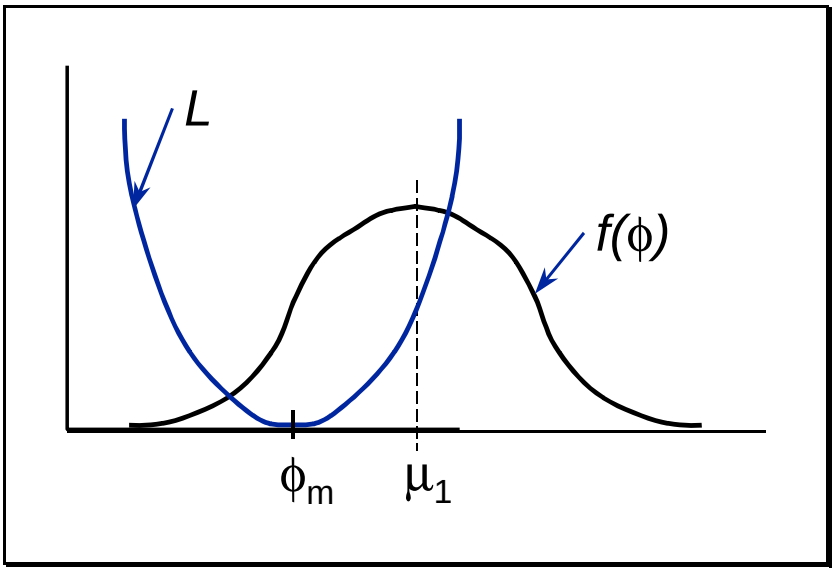


Figure 6-2: Combining the Loss Function with a Probability Distribution

Eq. 6-2 below shows how Eq. 6-1 expands out, and then simplifies using the basic definitions for a probability density function and the mean of a distribution.

$$\frac{L}{K} = \phi^2 f(\phi)d\phi - 2\phi_m \int \phi f(\phi)d\phi + \phi_m^2 \int f(\phi)d\phi \qquad \textbf{Eq. 6-2}$$

$$\frac{L}{K} = \phi^2 f(\phi)d\phi - 2\phi_m(\mu_1) + \phi_m^2(1)$$

Where:

$$1 = \int f(\phi)d\phi$$

$$\mu_1 = \int \phi f(\phi)d\phi$$

The expression can be further simplified by first adding and subtracting the same expression and then collecting terms as shown below in Eq. 6-3.

$$\frac{L}{K} = \phi^2 f(\phi)d\phi - 2\phi_m\mu_1 + \phi_m^2 + 2\mu_1^2 - 2\mu_1^2 \qquad \textbf{Eq. 6-3}$$

$$\frac{L}{K} = \phi^2 f(\phi)d\phi + \left(\mu_1^2 - 2\phi_m\mu_1 + \phi_m^2\right) + \mu_1^2 - 2\mu_1^2$$

$$\frac{L}{K} = \phi^2 f(\phi)d\phi + (\mu_1 - \phi_m)^2 + \mu_1^2 - 2\mu_1^2$$

Then using the same definitions from Eq. 6-2 the expression can be expanded, and then similar terms collected. The resulting expression (Eq. 6-1 below) is then simplified to a simple function of the variance, mean, and minimum cost point for the contact angle (or any other assembly function).

$$\frac{L}{K} = \phi^2 f(\phi)d\phi + (\mu_1 - \phi_m)^2 + \mu_1^2\left(f(\phi)d\phi\right) - 2\mu_1\left(\phi f(\phi)d\phi\right) \qquad \textbf{Eq. 6-4}$$

$$\frac{L}{K} = \left[\phi^2 - 2\mu_1\phi + \mu_1^2\right] f(\phi)d\phi + (\mu_1 - \phi_m)^2$$

$$\frac{L}{K} = (\phi - \mu_1)^2 f(\phi)d\phi + (\mu_1 - \phi_m)^2$$

$$\frac{L}{K} = \mu_2 + (\mu_1 - \phi_m)^2$$

Where: $\mu_2 = (\phi - \mu_1)^2 f(\phi)d\phi$

Thus the quality loss as defined by a quadratic loss function, where there is a minimum cost point is only a function of the distribution variance, mean, and that minimum cost point. That simplified equation for the quality loss is shown below in Eq. 6-5.

$$L = K\mu_2 + K(\mu_1 - \phi_m)^2 \qquad \textbf{Eq. 6-5}$$

For other loss functions (non-symmetrical, non-quadratic, etc.) Eq. 6-1 can be integrated numerically. For quadratic loss functions where the larger the assembly function the lower the cost, no finite minimum cost point, Creveling derives the simplified expressions [Creveling 1997, 214–218].

In terms of the one-way clutch assembly, the quality loss function might be estimated from the rework cost required at the specification limits. If one point on the cost curve is known (along with the minimum cost point), the cost constant *K* can be determined. For the one-way clutch assembly, if the rework cost at the specification limit (Δ = 7.0784 – 7.0184) is $20, K may be estimated as shown in Eq. 6-6 below.

$$L(\Delta) = K\Delta^2 \qquad \textbf{Eq. 6-6}$$

$$K = \frac{L(\Delta)}{\Delta^2} = \frac{\$20}{(0.60\,\text{deg})^2} = \$55.56/\text{deg}^2$$

Using Eq. 6-2 and Eq. 6-6, along with the estimation for the mean and variance of the contact angle, the quality loss can be estimated. Reductions in variance can be weighed against the quality savings. As was shown above, the quality loss is only a function of the mean and variance; the higher moments of the output distribution are not necessary.

Section 6.2. Design of Experiments in General

Design of experiments is a method of studying variation in a system without needing a mathematical expression to describe it. Variables that affect the system are the experimental variables or factors, and a measurement of the system is the response. The experimental variables are set to specific values and the response is measured.

Analyzing the average response for the levels of each of the variables yields the contribution to variance for each of the variables and interactions. The total variation of the response can estimate the true variation of the system or a percent contribution for the experimental variables. Design of experiments can be used to estimate variation in an existing process or system, or a mathematical model of a system.

Section 6.3. Design of Experiments in Variation Analysis

In the case of variation analysis, the experimental variables are the independent dimensions, and the response is the assembly function. The independent dimensions are set to specific values which are small perturbations around their means. The parts can then be made precisely to those dimensions and physically assembled, or the assembly function can be found (explicit or implicit) and the function calculated.

Physical parts must be made to a higher degree of precision than actual production parts, since the prescribed dimensions must be at the high or low end of the tolerance range in

various combinations. Thus, you cannot rely on the natural variation of production processes to produce such specific variations. Such precision requires costly processes similar to those used to make fixtures and production tooling.

Physically making the parts may be expensive to do, but it can provide additional information that a geometric model can not. The physical assemblies can be tested to determine the effects of noise (temperature, wear, misuse, etc.) [Taguchi 1986, 75]. The design of the assembly can thus be made more robust (less sensitive to noise and other variations). This study will not investigate the effects of noise, but is restricted to the mathematical modeling of the geometric variation in an assembly. The explicit equation for the contact angle of the clutch assembly will be used as the example.

Section 6.4. Two-level Designs

Generally, the more experimental runs, and the more levels of the independent variables, the greater and more reliable the information that the experiment can provide. Variation analysis is generally performed with either two or three-level designs. For a two-level design, each of the experimental variables is set above and below the mean values in specific combinations. A “run” is a measurement (or calculation) of the assembly function with all of the input variables set to a specific combination of their respective levels. A graphical representation of two-level designs for a three-factor problem is shown below in Figure 6-1.

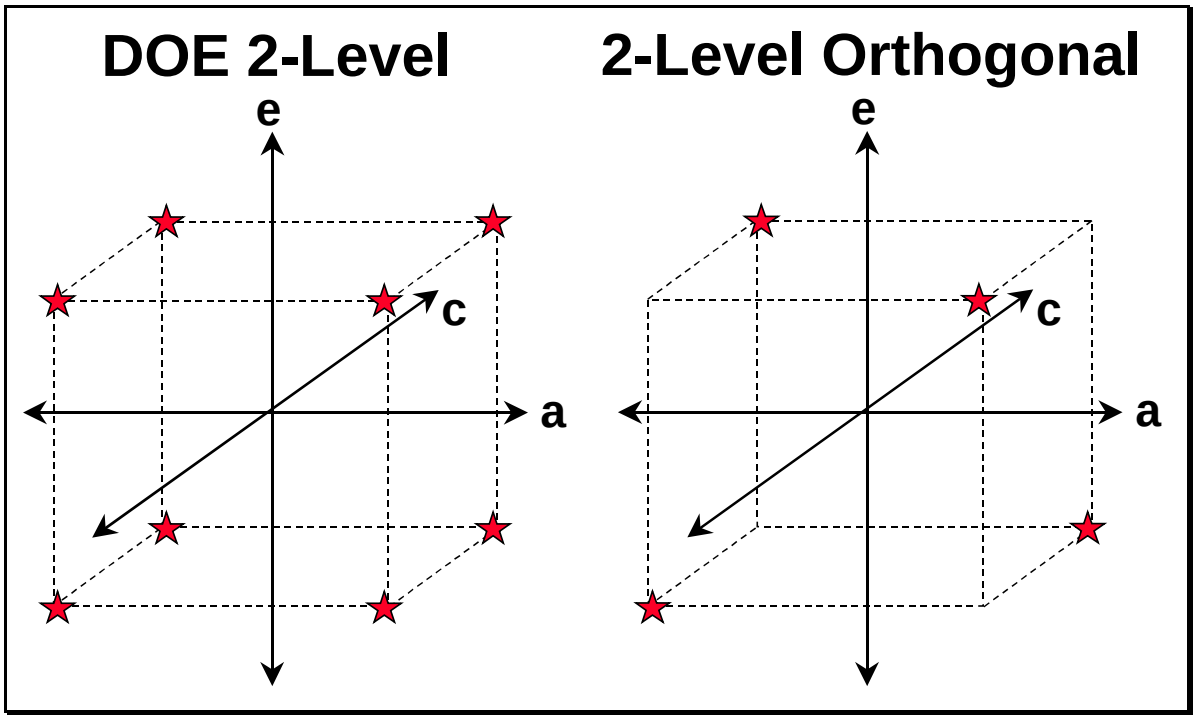


Figure 6-1: Two-level Experimental Designs for Three Factors

A complete two-level design requires $2^p = 8$ runs (where p is the number of factors). A two-level orthogonal design can have significantly fewer runs, depending on the desired information for the interactions. In general, a two-level orthogonal design requires about (p+1) runs. Orthogonal arrays have many uses, but their accuracy is difficult to predict. Other sources can provide more information on their use [Creveling 1997, 253–258; Taguchi 1986; Taguchi 1993].

Table 6-1: The Required Number of Runs for Typical Two-Level Designs

Variables	Number of Runs	
	Full Fact.	Orthogonal
p	2^p	(p+1)
3	8	4
7	128	8
11	2,048	12
15	32,768	16
31	2,147,483,648	32

An orthogonal design requires much fewer runs than a full factorial design, particularly as the number of variables increases. If the experimenting were to be performed by physically building the parts and assembling and testing them, an orthogonal array would save time and money. It would not be practical to physically build parts for a full factorial design for an assembly of more than seven independent variables being studied. But understanding the confounding (combining of the factor effects with interactions) that occurs with orthogonal designs can be difficult.

In order to illustrate how to analyze assembly variations using a two-level experimental design, a full two-level design will be analyzed for the clutch assembly. For a two-level design, the sampling points for the factors are plus and minus one standard deviation from the mean. Table 6-2 below shows the setup of the two-level design: the factors, the levels, and the interactions. In the table a +1 indicates that the dimension is at the mean value plus one standard deviation, a –1 means the mean minus one standard deviation. At these levels, the input variables are modeled as uniform distributions with the same mean and variance as their actual distributions. In the first three columns, every possible combination of the levels is included in a run.

Table 6-2: Two-level Experimental Design for the Clutch

Run	a	c	e	ac	ae	ce	ace
1	-1	-1	-1	+1	+1	+1	-1
2	-1	-1	+1	+1	-1	-1	+1
3	-1	+1	-1	-1	+1	-1	+1
4	-1	+1	+1	-1	-1	+1	-1
5	+1	-1	-1	-1	-1	+1	+1
6	+1	-1	+1	-1	+1	-1	-1
7	+1	+1	-1	+1	-1	-1	-1
8	+1	+1	+1	+1	+1	+1	+1

The ac, ae, ce, and ace columns represent the interactions of the variables. The levels for these interactions are determined by multiplying the values of the factors. For example, the level of ace = (a)(c)(e). In the analysis, the variations for the interaction terms are calculated just like the variations for the main factor.

After the experiment is set up the experimental runs can be performed. If physical parts are going to be made, the runs should be randomized in order to help prevent unplanned physical phenomena from skewing the results. Table 6-3 below shows the values of the independent variables for each of the runs, the value of the resulting contact angle ϕ, and the response (difference between the contact angle and the optimum contact angle of 7.0184 degrees). It is the response that will be analyzed to help determine the quality loss (the minimum loss is at the optimum contact angle).

Table 6-3: Two-level Experimental Results for the Clutch

Run	a	c	e	ϕ	Response (Change from Optimum)
1	27.6283	11.4267	50.7958	7.24317	**0.22477**
2	27.6283	11.4267	50.8042	7.33796	**0.31956**
3	27.6283	11.4333	50.7958	7.08820	**0.06980**
4	27.6283	11.4333	50.8042	7.18507	**0.16667**
5	27.6617	11.4267	50.7958	6.84766	**-0.17074**
6	27.6617	11.4267	50.8042	6.94790	**-0.07050**
7	27.6617	11.4333	50.7958	6.68351	**-0.33489**
8	27.6617	11.4333	50.8042	6.78621	**-0.23219**

The contact angle is calculated with Eq. 2-1. The response is the difference between the contact angle and the optimum contact angle (7.0184 degrees). The variation for each of the factors (including interactions) is found by summing the response for each level of each factor, and using Eq. 6-1 below [Taguchi 1993, 307].

$$Linear\,Variation_a = \frac{(\quad a_{+1} - \quad a_{-1})^2}{r(\lambda^2 S)} \qquad \textbf{Eq. 6-1}$$

$$Quadratic\,Variation_a = \frac{(\quad a_{+1} - 2 \quad a_0 + \quad a_{-1})^2}{r(\lambda^2 S)}$$

Where: a_{+1} = A value of the response with factor *a* at the +1 level

a_{-1} = A value of the response with factor *a* at the -1 level

r = The number of values summed in each sum effect

$\lambda^2 S$ = The sum of the squares of the coefficients of the sum effects (2 for the linear variation and 6 for the quadratic variation)

The calculation of the sum effects and variations for each of the factors is shown below in Table 6-4. It is clear that factor ***a*** has the most variation.

Table 6-4: Calculations of the Variations for the Two-level Design

Factor	Level	Sum Effect	Variation
a	-1	0.78081	**0.315663**
	+1	-0.80831	
c	-1	0.30310	**0.050196**
	+1	-0.33060	
e	-1	-0.21105	**0.019464**
	+1	0.18355	
ac	-1	-0.00475	**4.05E-05**
	+1	-0.02275	
ae	-1	-0.01940	**1.59E-05**
	+1	-0.00810	
ce	-1	-0.01602	**2.59E-06**
	+1	-0.01148	
ace	-1	-0.01394	**1.83E-08**
	+1	-0.01356	

The variations are calculated using Eq. 6-1. In this case r = 4, for four terms are summed into each one of the effects. All of the variations are linear because this experiment is only a two-level design. Eq. 6-2 below shows how the variation associated with the mean is calculated.

$$Variation_{mean} = \frac{\left(\sum_{i=1}^{n} response_i \right)^2}{n} \qquad \textbf{Eq. 6-2}$$

This variation affects the quality loss, and describes how far the mean contact angle is from the optimum. Table 6-5 below sums up the results from the two-level design of experiment for the clutch assembly.

Table 6-5: Summary Table for Two-level Clutch Experiment

Source	df	Variation	V/df	%Contribution	%Contribution without Mean
Mean	1	9.40E-05	9.40E-05	**0.024%**	-
a linear	1	0.315663	0.315663	**81.89%**	**81.909%**
c linear	1	0.050196	0.050196	**13.02%**	**13.025%**
e linear	1	0.019464	0.019464	**5.049%**	**5.0506%**
ac	1	4.05E-05	4.05E-05	**0.010%**	**0.0105%**
ae	1	1.59E-05	1.59E-05	**0.004%**	**0.0041%**
ce	1	2.59E-06	2.59E-06	**0.001%**	**0.0007%**
ace	1	1.83E-08	1.83E-08	**0.000%**	**0.0000%**
error	0	0	-	-	-
Total	8	0.38547685	**0.04818461**	100%	100%
Less Mean Variance			**0.00001175**		
Pure Variance (no mean error)			**0.04817286**		

The pure variance is calculated by subtracting the mean variance from the total variance and dividing by the degrees of freedom. The number of degrees of freedom is one for each of the main effects and interactions, and the total degrees of freedom is the total number of runs. The number of degrees of freedom for the error is the difference between the total and the main effects and interactions.

The percent contribution without the mean is comparable to the percent contribution obtained by the other methods (Monte Carlo, method of system moments, etc.) The mean and the standard deviation for the contact angle can be calculated directly from the calculations for the contact angle for the runs in Table 6-3. These values will be summarized with the others at the end of this chapter.

Section 6.5. Three-level Designs

Three-level designs are similar to two-level designs, except that the factors are also studied at their mean values. Figure 6-1 below shows a full three-level design and an orthogonal design. The three-level design calculation for the clutch assembly will be shown for the orthogonal design.

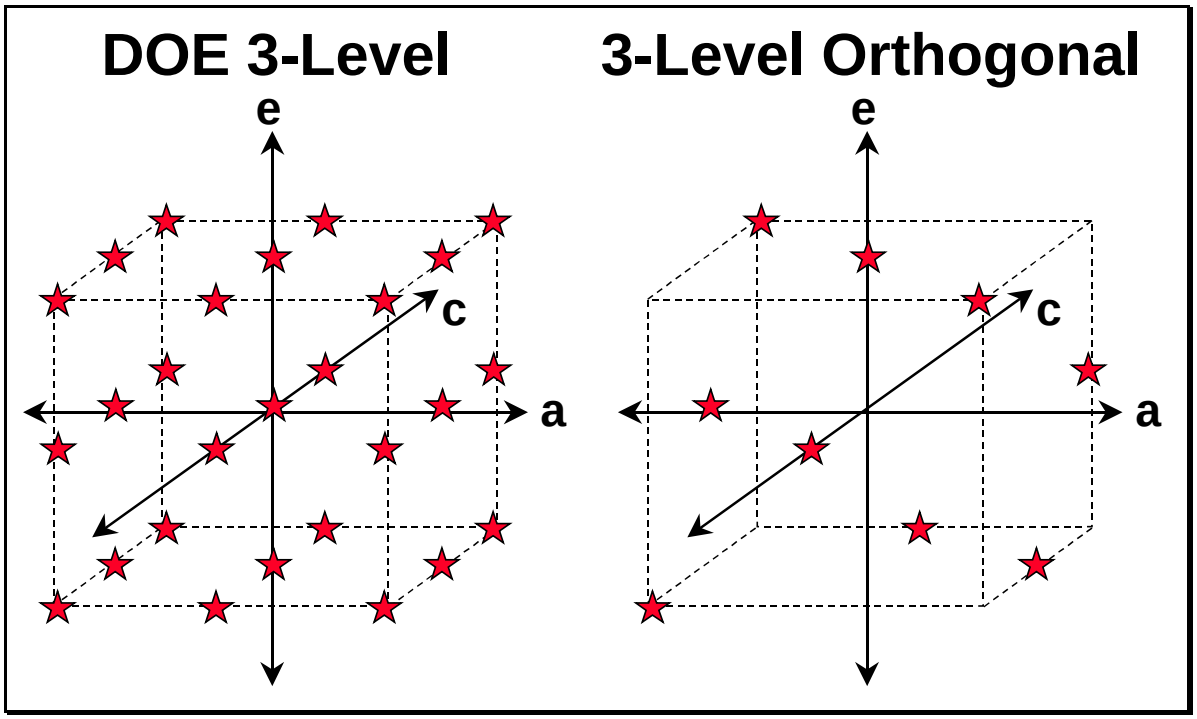


Figure 6-1: Three-level Experimental Designs for Three Factors

A complete three-level design requires $3^P = 27$ runs (where p is the number of factors). Problems with larger numbers of factors would thus require many runs. An orthogonal array requires only a fraction of the full factorial [Creveling 1997, 253–258;Taguchi 1986; Taguchi 1993].

Table 6-1: Number of Runs for Common Three-Level Orthogonal Designs

	Number of Runs	
Variables	Full Fact.	Orthogonal
p	3^p	3(p-1)
4	81	9
13	1,594,323	36

There are not orthogonal designs tailored for all different numbers of variables, but an orthogonal design for more variables can always be used for less variables (the extra columns would just be wasted or used for interactions).

The design of the experiment, the values for the factors, and the calculated responses for the clutch assembly are shown below in Table 6-2. The interaction term will not be studied with the orthogonal design, because of the confounding of their effects with the main effects. The values for each of the factors in this Taguchi-type three-level design are the mean (level 0), $+\sigma\sqrt{3/2}$ (level +1), and $-\sigma\sqrt{3/2}$ (level –1). At these levels, the input variables are modeled as uniform distributions with the same mean and variance as their actual distributions.

Table 6-2: Three-level Experimental Design for the Clutch Assembly

	Factor Level			Factor Value				Response (Change from Optimum)
Run	**a**	**c**	**e**	**a**	**c**	**e**	**ϕ**	
1	-1	-1	-1	27.6246	11.4259	50.7949	7.29274	**0.27434**
2	-1	0	0	27.6246	11.4300	50.8000	7.25746	**0.23906**
3	-1	+1	+1	27.6246	11.4341	50.8051	7.22201	**0.20361**
4	0	-1	0	27.6450	11.4259	50.8000	7.11461	**0.09621**
5	0	0	+1	27.6450	11.4300	50.8051	7.07845	**0.06005**
6	0	+1	-1	27.6450	11.4341	50.7949	6.85935	**-0.15905**
7	+1	-1	+1	27.6654	11.4259	50.8051	6.93197	**-0.08643**
8	+1	0	-1	27.6654	11.4300	50.7949	6.70808	**-0.31032**
9	+1	+1	0	27.6654	11.4341	50.8000	6.66973	**-0.34867**

Even for three levels, the orthogonal design requires only nine runs. The response is the difference between the contact angle and the optimal angle of 7.0184 degrees. This orthogonal three-level design is as easy to perform as the full two-level design (only one more run). Because this is a three-level design, the quadratic variations for the factors can be estimated, although they are small relative to the linear variations. The variations for the interactions are pooled into the error term. Table 6-3 below shows the calculations for the linear and quadratic variations, and Table 6-4 shows the summary table for the variations of the factors, error, and mean.

Table 6-3: Calculations for the Orthogonal Three-level Design

Factor	Level	Sum Effect	Linear Variation	Quadratic Variation
a	-1	0.71701	**0.356454**	**2.90E-05**
	0	-0.00279		
	+1	-0.74543		
c	-1	0.28412	**0.057669**	**3.28E-07**
	0	-0.01121		
	+1	-0.30411		
e	-1	-0.19503	**0.023096**	**4.50E-06**
	0	-0.01340		
	+1	0.17723		

The calculations for the linear and quadratic variations are made using Eq. 6-1, with r = 3. Table 6-4 below summarizes the estimation of the total variation and percent contribution for the three-level orthogonal design.

Table 6-4: Summary Table for the Three-level Orthogonal Design

Source	df	Variation	V/df	%Contribution	%Contribution without Mean
Mean	1	1.08E-04	1.08E-04	**0.025%**	-
a (linear)	1	0.356454	0.356454	**81.50%**	**81.519%**
aa (quad.)	1	2.90E-05	2.90E-05	**0.007%**	**0.0066%**
c (linear)	1	0.057669	0.057669	**13.19%**	**13.189%**
cc (quad.)	1	3.28E-07	3.28E-07	**0.000%**	**0.0001%**
e (linear)	1	0.023096	0.023096	**5.281%**	**5.2819%**
ee (quad.)	1	4.50E-06	4.50E-06	**0.001%**	**0.0010%**
error	2	1.03E-05	5.14E-06	**0.002%**	**0.0024%**
Total	9	0.43737135	**0.04859682**	100%	100%
Less Mean Variance			**1.20E-05**		
Pure Variance (no mean error)			**0.04858480**		

The size of the error term indicates that the variations for the interactions not studied are relatively small. If the error term were large relative to the others, a better design would be chosen that could study the interactions, and the study would be redone.

Section 6.6. Three-Level Weighted Design

An alternate three-level design, in which the factors at each level are weighted to simulate a normal distribution for the input variable distributions, can be used to better estimate the skewness and kurtosis [Creveling 1997, 292; D'Errico 1988]. The weights approximate the probability of that level of the factor occurring. Figure 6-1 shows these weights and the sampling points for the factors.

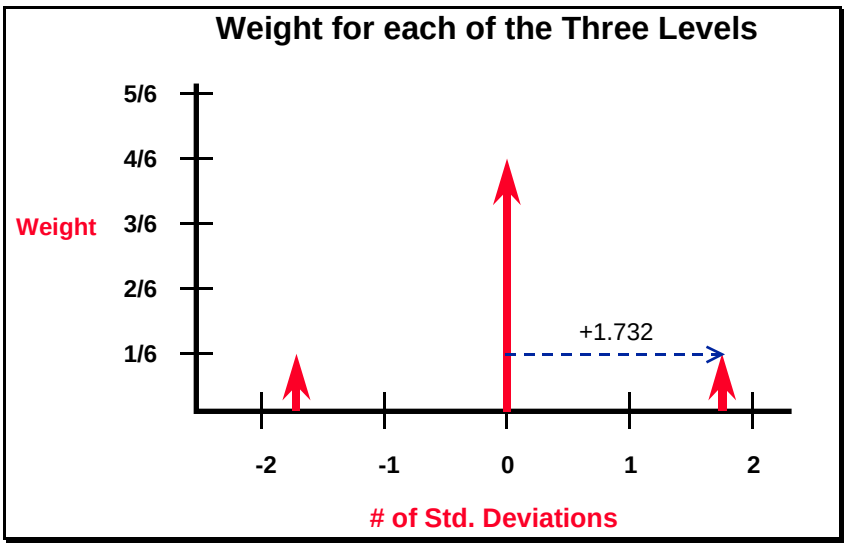


Figure 6-1: Weightings for the Alternate Three-Level Design

The mean levels for this alternate design are weighted 2/3, and the upper and lower levels (at $\pm\sigma\sqrt{3}$) are each weighted 1/6. The levels for this design are based on a quadrature formula used for numerical integration demonstrated by Evans [Evans 1967; Evans 1971]. This weighting approximates the input variables as normally distributed. Calculating the first four moments for the input distributions of the factors using the weighting above results in normal distributions, as shown below in Eq. 6-1. The methods presented earlier yield input distribution moments for a uniform distribution (only the kurtosis is different than a normal distribution).

$$\mu_1 = \frac{1}{6}\left(\mu_1 - \sigma\sqrt{3}\right) + \frac{4}{6}(\mu_1) + \frac{1}{6}\left(\mu_1 + \sigma\sqrt{3}\right) = \mu_1 \qquad \textbf{Eq. 6-1}$$

$$\mu_2 = \frac{1}{6}\left(\mu_1 - \sigma\sqrt{3} - \mu_1\right)^2 + \frac{4}{6}(\mu_1 - \mu_1)^2 + \frac{1}{6}\left(\mu_1 + \sigma\sqrt{3} - \mu_1\right)^2 = 2\frac{\sigma^2 3}{6} = \sigma^2$$

$$\mu_3 = \frac{1}{6}\left(\mu_1 - \sigma\sqrt{3} - \mu_1\right)^3 + \frac{4}{6}(\mu_1 - \mu_1)^3 + \frac{1}{6}\left(\mu_1 + \sigma\sqrt{3} - \mu_1\right)^3 = (0)\frac{\sigma^3 3\sqrt{3}}{6} = 0$$

$$\mu_4 = \frac{1}{6}\left(\mu_1 - \sigma\sqrt{3} - \mu_1\right)^4 + \frac{4}{6}(\mu_1 - \mu_1)^4 + \frac{1}{6}\left(\mu_1 + \sigma\sqrt{3} - \mu_1\right)^4 = 2\frac{\sigma^4 9}{6} = 3\sigma^4$$

To perform the experiment with the weighted levels, each run is weighted as the product of the weights of the factors (determined by the levels for each factor). Table 6-1 below shows the first twelve of the twenty-seven runs for this weighted design. The weights for each of the factors are shown, as well as the resulting weight of the run (product of the weights for each of the factors) [Creveling 1997, 295].

Table 6-1: First Twelve (of Twenty-Seven) Runs for the Weighted Three-Level Design

	Level			Factor Weights			Run	Factor Values			Response
Run	**a**	**c**	**e**	**a**	**c**	**e**	**Weight**	**a**	**c**	**e**	**ϕ**
1	-1	-1	-1	1/6	1/6	1/6	1/216	27.6161	11.4242	50.7928	**7.403427**
2	-1	-1	0	1/6	1/6	4/6	4/216	27.6161	11.4242	50.8000	**7.483810**
3	-1	-1	1	1/6	1/6	1/6	1/216	27.6161	11.4242	50.8072	**7.563316**
4	-1	0	-1	1/6	4/6	1/6	4/216	27.6161	11.4300	50.7928	**7.272379**
5	-1	0	0	1/6	4/6	4/6	16/216	27.6161	11.4300	50.8000	**7.354223**
6	-1	0	1	1/6	4/6	1/6	4/216	27.6161	11.4300	50.8072	**7.435142**
7	-1	1	-1	1/6	1/6	1/6	1/216	27.6161	11.4358	50.7928	**7.138899**
8	-1	1	0	1/6	1/6	4/6	4/216	27.6161	11.4358	50.8000	**7.222284**
9	-1	1	1	1/6	1/6	1/6	1/216	27.6161	11.4358	50.8072	**7.304693**
10	0	-1	-1	4/6	1/6	1/6	4/216	27.6450	11.4242	50.7928	**7.069906**
11	0	-1	0	4/6	1/6	4/6	16/216	27.6450	11.4242	50.8000	**7.154080**
12	0	-1	1	4/6	1/6	1/6	4/216	27.6450	11.4242	50.8072	**7.237249**

The response function is the variation of the contact angle (instead of the difference from the optimum contact angle) for the clutch assembly. Weighting the responses will give a good estimate of the output distribution for the contact angle. To calculate the moments for the contact angle with the weighted experimental design, Eq. 6-2 should be used.

Creveling indicates that the full 3^p factorial array (or all twenty-seven runs in this case) should be used for correct statistical calculations [Creveling 1997, 294].

Eq. 6-2

$$\mu_1 = \frac{\sum_{i=1}^{n} w_i \phi_i}{\sum_{i=1}^{n} w_i} \qquad \mu_3 = \frac{\sum_{i=1}^{n} w_i (\phi_i - \mu_1)^3}{\sum_{i=1}^{n} w_i}$$

$$\mu_2 = \frac{\sum_{i=1}^{n} w_i (\phi_i - \mu_1)^2}{\sum_{i=1}^{n} w_i} \qquad \mu_4 = \frac{\sum_{i=1}^{n} w_i (\phi_i - \mu_1)^4}{\sum_{i=1}^{n} w_i}$$

Where: w_i = The weight of the i^{th} run (product of the weights for the factors for the i^{th} run)

Using the above equations to calculate the first four moments for the contact angle output distribution yields the results in Table 6-2 below.

Table 6-2: Comparison of Weighted DOE with Monte Carlo

Contact Angle for the Clutch	Monte Carlo	Weighted DOE	%Error
μ1 (mean).....	7.014953	7.014956	0.0000%
σ (Standard Deviation).....	0.219668	0.219669	0.0005%
α3 (Skewness).....	-0.094419	-0.094031	-0.4114%
α4 (Kurtosis).....	3.023816	3.012274	-0.3817%
Quality Loss ($/part).....	2.681	2.681	0.0010%
Lower Rejects (ppm).....	4,406	4,238	-3.8011%
Upper Rejects (ppm).....	2,166	2,353	8.6210%
Total Rejects (ppm).....	6,572	6,591	0.2931%

The rejects for the weighted, three-level design were calculated by fitting a Lambda distribution to the distribution moments and applying the specification limits for the contact angle. The errors for the weighted DOE are very small, most even smaller than the second order method of system moments (refer to Table 4-1). But the weighted three-level design requires that the input variables are normally distributed, and the number of runs required escalates greatly with the number of independent variables (10 variables would require 59,049 runs).

Section 6.7. Sample Clutch Problem DOE Results

Five experimental designs were presented in the sections above: two-level (full and orthogonal), three-level (full and orthogonal), and weighted three-level designs. The designs were executed by using the explicit equation for the contact angle of the clutch assembly (see Eq. 2-1). The results for these experimental designs are shown below in Table 6-1 and Table 6-2.

Table 6-1: Summary of Moment Results from DOE

Contact Angle	Monte Carlo	2-Level	2-Level Orthogonal	3-Level	3-Level Orthogonal	3-Level Weighted
Distribution.....	**Normal**	**Uniform**	**Uniform**	**Uniform**	**Uniform**	**Normal**
$\mu 1$.....	7.014953	7.014962	7.014914	7.014961	7.014933	7.014956
σ.....	0.219668	0.219483	0.221007	0.219530	0.220420	0.219669
$\alpha 3$.....	-0.094419	-0.029201	-0.460403	-0.045394	-0.296061	-0.094031
$\alpha 4$.....	3.023816	1.619923	1.638456	1.967458	1.650788	3.012274
Quality Loss....	2.681	2.677	2.714	2.678	2.700	2.681

The “distribution” row of the table above indicates how the input variables are modeled. The distribution type heavily affects the skewness and kurtosis of the output distribution. The only experimental design that is actually designed to estimate the third and fourth moments of the output distribution is the three-level weighted design (if the input variables are normally distributed). All of the methods estimate the mean, standard deviation, and the quality loss quite well.

Table 6-2: Summary of Percent Contribution Results from DOE

Contact Angle	2-Level	2-Level Orthogonal	3-Level	3-Level Orthogonal	3-Level Weighted
a	81.909%	81.246%	81.907%	81.519%	81.902%
aa	NA	NA	0.008%	0.007%	0.033%
c	13.025%	13.308%	13.020%	13.189%	13.007%
cc	NA	NA	0.000%	0.000%	0.001%
e	5.051%	5.446%	5.049%	5.282%	5.043%
ee	NA	NA	0.000%	0.001%	0.000%
ac	0.011%	NA	0.011%	NA	0.011%
ae	0.004%	NA	0.004%	NA	0.004%
ce	0.001%	NA	0.001%	NA	0.001%
ace	0.000%	NA	0.000%	NA	0.000%
other	NA	NA	0.000%	0.002%	0.000%
Total	100%	100%	100%	100%	100%

The percent contributions above are estimated without including the variation of the mean from the target. The factor “a” represents the linear percent contribution of “a,” and the factor “aa” represents the quadratic percent contribution of “a.” All of the methods yield similar estimates for the contribution of the linear effects. All of the other effects are quite small (quadratic and interaction). The percent contributions for the quadratic effects are just like the linear effects as they sum up to the total. The percent contribution estimate from the three-level weighted design was difficult to perform because of the weightings of the levels and runs. That design is not generally used to estimate the percent contribution, but rather the overall distribution moments.

Not all of the DOE methods are designed to estimate the analysis objectives of skewness, kurtosis, and percent contribution, but where possible the estimates were calculated. For example, all of the experimental designs, except for the three-level weighted design, model the input variables as uniform distributions. Thus, the estimates for skewness and kurtosis of the output distribution are biased toward that of a uniform distribution. The more factors in an assembly, the closer the response will be to a normal distribution.

Since the clutch has only three factors, the kurtosis is still near 1.8. Both the normal and the uniform distributions have zero skewness.

Section 6.8. Response Surfaces

Another application for design of experiments, besides looking at the variation of the output function, is to create a response surface from the experimental runs. Response surfaces are generally used to study a large region of a surface, not a small region as with variation analysis. The response surface can used to find optimum design points where the sensitivity of the assembly function with respect to input variations is minimized. Instead of modifying the variations of the inputs, the nominal values are modified to determine a more robust design.

Generally, a linear or a quadratic surface is fit to the data by least-squares regression. This section will focus on the use of quadratic surfaces. Figure 6-1 below shows two popular quadratic response surface designs.

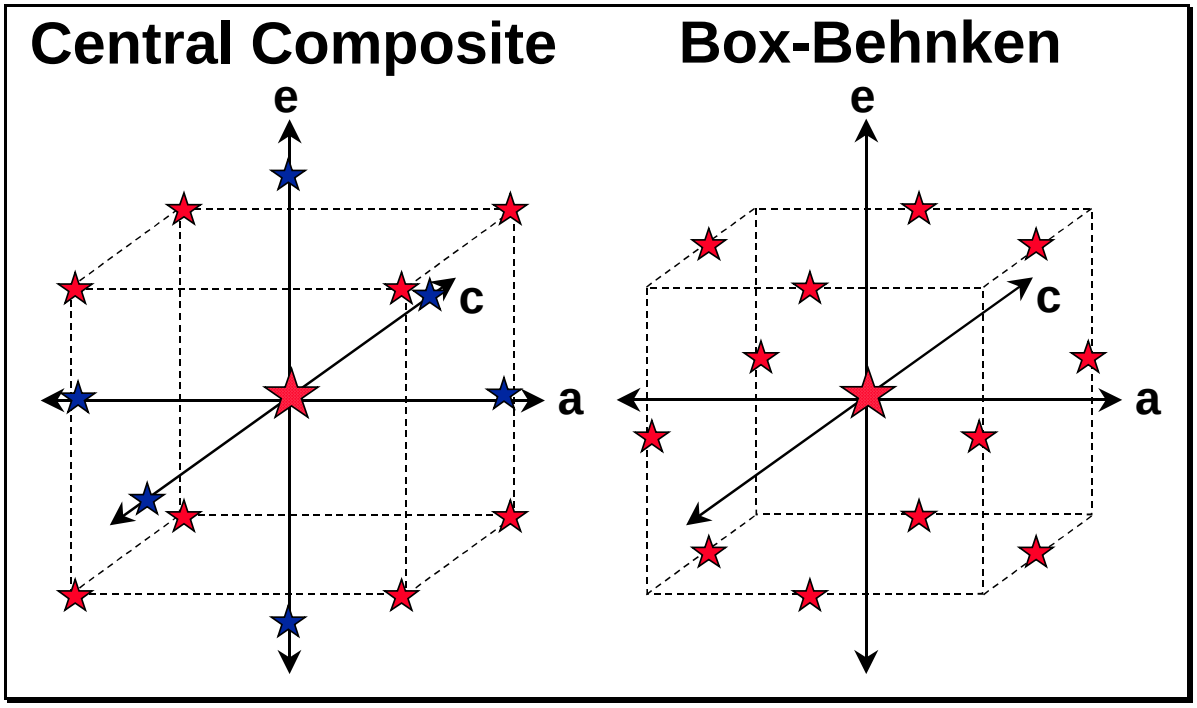


Figure 6-1: Popular Response Surface Designs

The central composite design (CCD) consists of a two-level factorial design (points on the corner of the cube), plus axial points, plus points in the center. The Box-Behnken design (BBD) consists of the points on the edges of the cube plus points in the center.

The CCD is designed to be a step-wise design. The factorial points are run first (2^p runs), then the center point. The center point helps to determine if there is curvature in the response surface. If there is little curvature in the response surface, a linear response surface can then be fit to the data, otherwise the axial points can be run before fitting the full quadratic surface. The axial points can either be on the surface of the cube, or further out on the surface of a sphere through the corner points.

For physically building the assemblies, multiple runs are performed in the center. Having multiple points in the center helps to determine the pure experimental error (error inherent in the setup and measurement of the design).

The BBD consists of multiple center runs, like the CCD, but all of the other points are equidistant from the center (spherical design). And all of the levels for the factors are evenly spaced (for the CCD the levels for the factorial and axial points are not necessarily the same). The CCD, in contrast, can either be spherical or have evenly spaced levels, not both. The CCD better covers the entire space of the cube, especially with greater numbers of variables [Myers 1971; Myers 1995].

6.8.1. Using a Response Surface

Once the data is taken and a quadratic surface fit to assembly function, it can be used to help determine better designs. The derivatives of the response surface represent the sensitivities of the response to a variable at that point in the design space. Therefore, the design can be altered to reduce the sensitivity of the response to certain variables. For example, Figure 6-1 below shows a response surface for the contact angle.

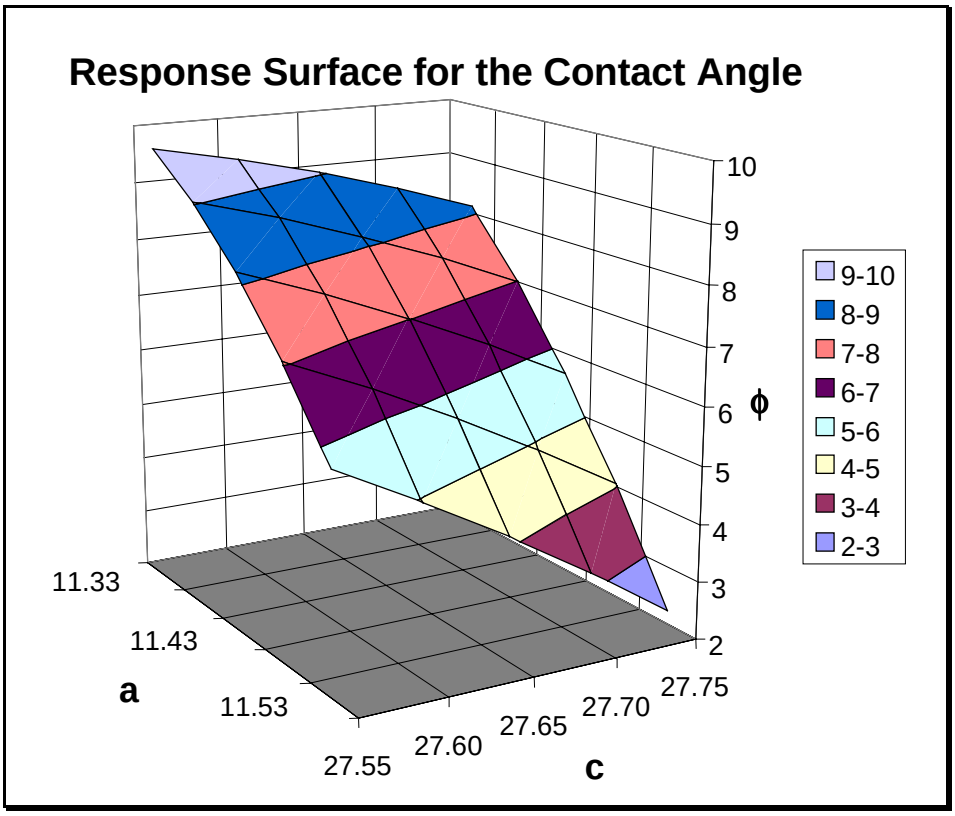


Figure 6-1: Response Surface for ϕ versus a and c

The quadratic response surface shown above was obtained through a face-centered central composite design. This graph was plotted holding the ring radius, **e**, constant. The combinations of values for the input variables for the optimum contact angle can thus be studied along with the sensitivities (partial derivatives). The quadratic equation represented in the figure above is shown below in Eq. 6-1.

$$\phi = 2{,}786.18 - 150.15a - 137.28c + 3.06e - 12.64a^2 - 44.57c^2 - 12.79e^2 - 48.19ac + 27.31ae + 48.50ce \qquad \textbf{Eq. 6-1}$$

With Eq. 6-1 above the sensitivities and percent contribution for the quadratic factors can all be estimated at any value of the input variables; all that it requires is to take the partial derivative of that equation with respect to the variables. Eq. 6-2 below shows the partial derivatives, and how they are calculated in terms of the values of the independent variables.

$$\frac{\partial \phi}{\partial a} = -150.15 - (2)12.64a - 48.19c + 27.31e = -12.48 \qquad \textbf{Eq. 6-2}$$

$$\frac{\partial^2 \phi}{\partial a^2} = -(2)12.64 = -25.28$$

$$\frac{\partial^2 \phi}{\partial a \partial c} = -48.19$$

$$\frac{\partial^2 \phi}{\partial a \partial e} = 27.31$$

The sensitivities above are easily calculated. Only the first derivative above is a function of the input variable values (evaluated at a = 27.645, c = 11.43, and e = 50.8). These sensitivities can be used to improve the design (changing the values of the inputs to reduce the sensitivities), or to estimate the percent contribution of each of the inputs to the variation in the assembly function. Chapter 12 will show how these sensitivities can be used to increase the accuracy of the method of system moments.

It appears from both Eq. 6-2 and Figure 6-1 that if the hub radius ***a*** were reduced, the sensitivity (partial derivative) of ϕ with respect to it would decrease. But the other variables would also have to adjust to maintain the optimum contact angle. Thus the product design can be modified to reduce the sensitivity of the performance with respect to variations in components.

Section 6.9. Summary for DOE and Response Surfaces

The major strengths of the design of experiments is the reduced number of trials as compared to Monte Carlo, the ease of understanding, and the ability to incorporate noise and other non-geometric variation into the model. Interactions between variables can also be determined, but require more complex experimental designs. The major weaknesses of design of experiments are understanding orthogonal arrays, the confounding of the interactions, and understanding the differences in the alternative methods of design of experiments.

The objectives of the designs are different. Only the three-level weighted design is really designed to determine the first four output distribution moments, where the others are designed to analyze the contributions to variance easily. Response surfaces are not designed to directly determine the variation of the output, but rather the shape and sensitivities of the output function over a region. Thus the design should be chosen first according to the objective, and then to the desired accuracy. Section 14.2 will how DOE and response surfaces compare to the other variation methods also according to objective and accuracy.

Chapter 7. Lambda Distribution

This chapter will discuss how the Lambda distribution can be used with variation analysis. The state-of-the-art for the Lambda distribution will be presented in terms of its use, calculation of the moments of a Lambda distribution, and determining the Lambda parameters to match a real distribution. This chapter will also make contributions for the Lambda distribution by presenting a method to more robustly determine the Lambda parameters to match a distribution, and by understanding the nature of the Lambda distribution better.

Section 7.1. Introduction to the Lambda Distribution

Fitted distributions are used to compute percent (or parts per million) rejects of the assembly objective by applying the specification limits and integrating the area outside the limits. Any distribution can be represented in terms of the moments of area about the centroid (or mean). If the moments of a distribution are known, a standard distribution can be fit to those moments and used to estimate percent rejects.

Several different distributions are commonly used in statistics, such as the normal, lognormal, Weibull, gamma, beta, uniform, etc. The number of parameters used in these analytical distributions determines how many moments of can be matched. Most of these distributions, particularly the normal distribution, are only two parameter models and therefore can match only two moments: the mean and the variance. To accurately represent non-normal distributions, a four-parameter model is required. The Lambda distribution has four parameters and can thus model a wide variety of distributions.

7.1.1. Background of the Lambda Distribution

The Lambda distribution is a general distribution that requires four parameters: λ_1, λ_2, λ_3, and λ_4. With these parameters, the Lambda distribution can model distributions with a

wide range of different characteristics. Thus, if four percentiles or the first four moments (μ_1, μ_2, μ_3, and μ_4) of a data set are known, the Lambda distribution can provide a good model for statistical analysis. The common equations useful for the Lambda distribution [Shapiro 1981, 181] are shown below as Eq. 7-1 and Eq. 7-2.

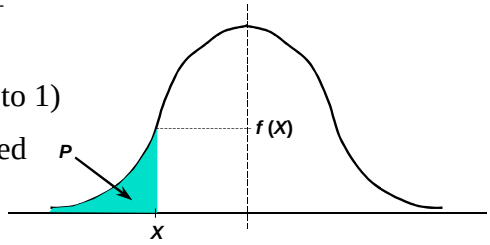
$$X = \lambda_1 + \frac{P^{\lambda_3} - (1-P)^{\lambda_4}}{\lambda_2} \qquad \textbf{Eq. 7-1}$$

$$f(X) = \frac{\lambda_2}{\lambda_3 P^{\lambda - 1} + \lambda_4 (1-P)^{\lambda_4 - 1}} \qquad \textbf{Eq. 7-2}$$

Where: P = Cumulative probability desired (0 to 1)

X = Value of the variable being modeled

$f(X)$ = Probability density function



To use Lambda distribution with the equations above, the X value must be obtained for a probability. Eq. 7-1 is very useful for generating random data for the Lambda distribution because the cumulative probability P would just be a uniform random number 0.0 to 1.0. Eq. 7-2 could be used in combination with Eq. 7-1 to graph the probability density function in terms of X.

If the cumulative probability for a certain X is desired, Eq. 7-1 must be solved through iteration. The method implemented for this study was the Regula Falsi method [Press 1992, 354] due to the non-convergence possibilities associated with the Newton-Raphson method. Of course, before the above equations can be used, the Lambda parameters must be chosen. Most of this chapter will focus on understanding the Lambda distribution and choosing the Lambda parameters to fit a distribution.

7.1.2. Comparison to other General Distributions

The Lambda distribution can model many different distributions. Figure 7-1 below shows several distributions characterized by α_3 and α_4.

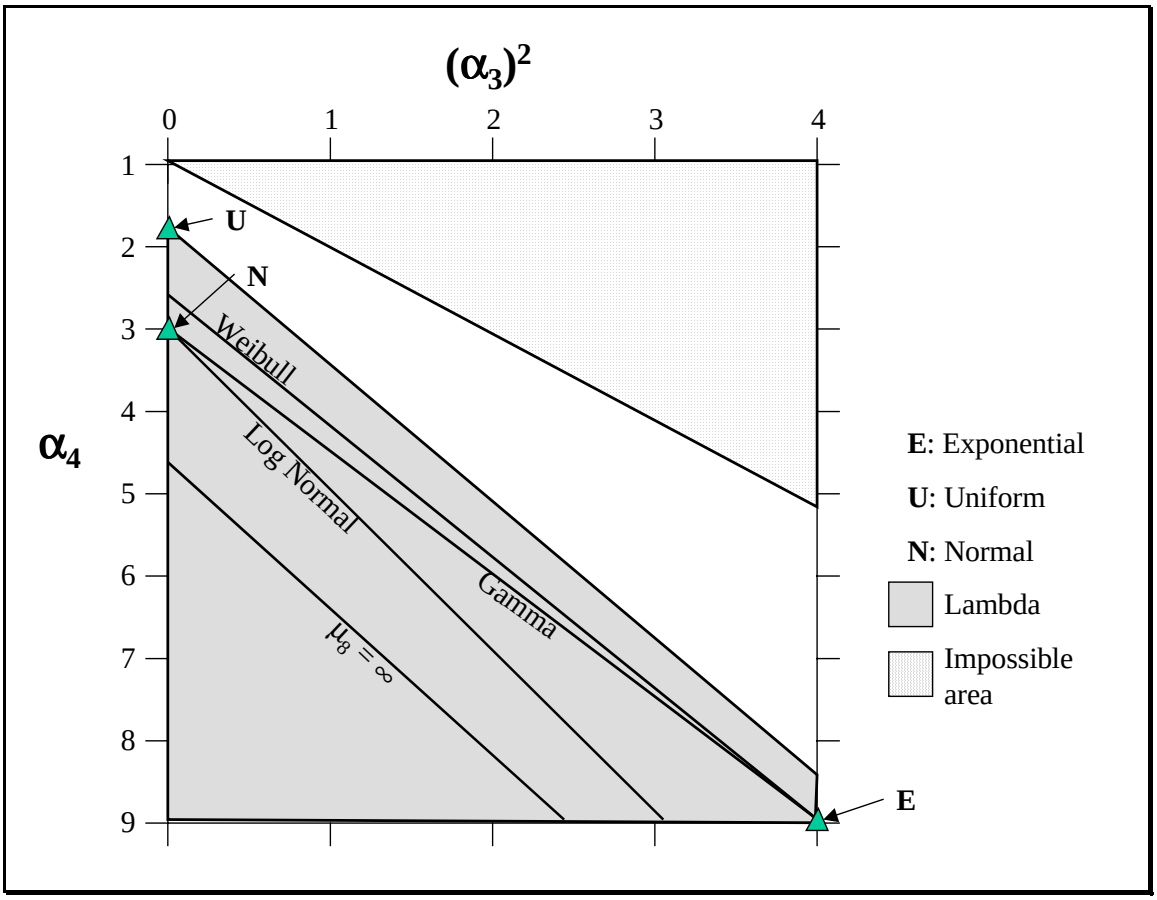


Figure 7-1: Comparison of Lambda to other Distributions

The Lambda distribution can model a wide variety of distributions, including the Normal, Uniform, and Exponential distributions. Some of the area not covered by the Lambda distribution could be covered by the Beta distribution or the Johnson distribution [Freimer 1988, 3561; Glancy 1994, 19; Ramberg 1979, 206].

Although the figure above indicates that the Lambda distribution covers the region of the Normal distribution, it does not model it exactly. The table below shows a comparison of the first eight moments of the standard Normal distribution compared to the corresponding Lambda distribution.

Table 7-1: Standard Normal vs. Lambda

μ	Std Normal	Lambda	%Error
1	0	0	0.00%
2	1.000	1.000	0.00%
3	0	0	0.00%
4	3.000	3.000	0.00%
5	0	0	0.00%
6	15.000	14.710	-1.93%
7	0	0	0.00%
8	105.000	97.647	-7.00%

The table above shows that the Lambda distribution can model the Normal distribution exactly through the first five moments (μ_1–μ_5), but the sixth and eight moments do not match perfectly. The error in parts per million rejects using the Lambda distribution versus the Normal distribution can be seen in Figure 7-2 below.

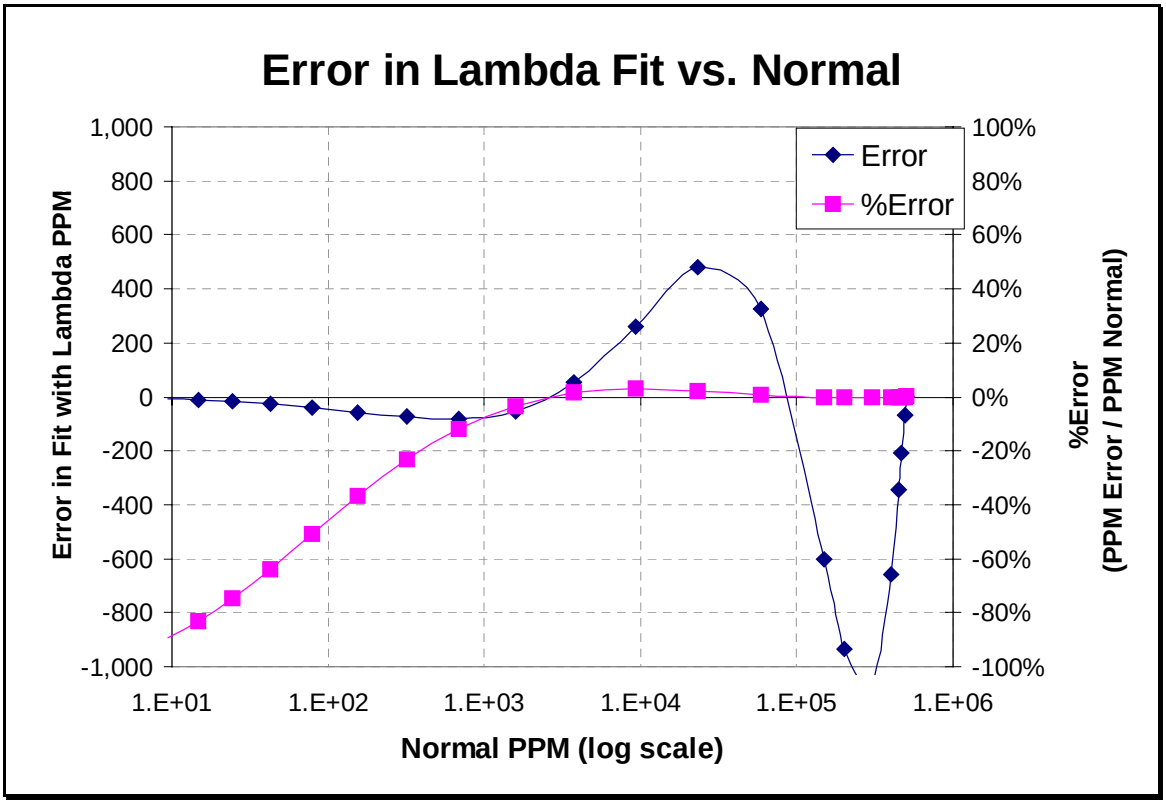


Figure 7-2: Error in Modeling a Normal distribution (left tail) with the Lambda

The percent error in parts per million rejects (PPM) for the Lambda distribution increases to about 90 percent as the normal PPM decreases to 10. The percent error for a normal PPM of 10,000 is about 3 percent. The Lambda distribution obtained by matching the first four moments of the normal distribution does not model the normal distribution well far out in the tails.

The Lambda distribution can be used to fit many different distributions or data sets. But, as was shown above, when using a known distribution (such as the Normal), the known distribution should be used. But when dealing with production processes, there is no such thing as a “known” distribution. All distributions are obtained by fitting the moments or percentiles calculated from the data sets. Often only the first few moments of a data set are calculated to see if they are close to normal. The higher moments could be quite different from normal, and assuming normality adds some uncertainty to the analysis from the start. Section 13.1 discusses how to evaluate the uncertainty of input variable information.

To say that the higher moments of the Lambda distribution are incorrect because they do not match the normal distribution is not true. To determine the accuracy of the higher moments, one would have to calculate the higher moments for the data set being fit and compare them to the Lambda distribution fit by the first four moments. Many natural processes tend to be normally distributed, but tending to be normal is not the same as being normal.

Section 7.2. Fitting the Lambda Distribution

The four parameters of the Lambda distribution are commonly determined by two different methods: matching percentiles and matching moments. Either of the two methods is valid, and the choice would depend on the type of information available and goals for the analysis [Ramburg 1979, 205; Shapiro 1981, 185].

7.2.1. Matching Percentiles

To match percentiles, four percentiles and corresponding *x* values from the data set are chosen. Eq. 7-1 is used four times, creating a system of four non-linear equations and four unknowns (λ_1 through λ_4). The system of non-linear equations must then be solved, resulting in the fit distribution.

Matching percentiles is a method that fits nicely with actual data sets. The percentiles of the data set to use for fitting would depend on the desired use for the Lambda distribution. If the behavior of the distribution in the tails is desired, than the percentiles closer to the tails should be used for the fit. But for small data sets extreme percentiles are very unreliable, and should be used with extreme caution.

7.2.2. Matching Moments

Matching moments is a good technique when the moments of a distribution are known or hypothetically chosen. This is normally the case when performing variation analysis by the Method of System Moments or Design of Experiments. Usually the first four moments of the distribution are used (mean, variance, skewness, and kurtosis). Often only the first two, or maybe three, moments of a distribution are known with any confidence.

The task of matching moments is to determine the values of λ_3 and λ_4 that result in the desired standardized skewness and kurtosis. λ_1 is a location parameter related mainly to the mean, and λ_2 is a scaling parameter similar to the variance. λ_3 and λ_4 together determine the standardized skewness and kurtosis of a distribution. To solve for λ_3 and λ_4, a system of two non-linear equations must be solved. The equations useful for matching the first four moments are shown below as Eq. 7-1 [Ramberg 1979, 204–205]. The more general equation for any moment of the Lambda distribution can be found in article by Ramberg [Ramberg 1979, 204] and the book by Shapiro [Shapiro 1981, 183].

$$\mu_1 = \lambda_1 + A/\lambda_2 \qquad \mu_2 = \sigma^2 = (B-A)^2/\lambda_2^{\,2} \qquad \textbf{Eq. 7-1}$$

$$\mu_3 = (C - 3AB + 2A^3)/\lambda_2{}^3 \qquad \mu_4 = (D - 4AC + 6A^2B - 3A^4)/\lambda_2{}^4$$

$$\alpha_3 = \mu_3/\sigma^3 \qquad \alpha_4 = \mu_4/\sigma^4$$

Where:

$$A = 1/(1+\lambda_3) - 1/(1+\lambda_4)$$

$$B = 1/(1+2\lambda_3) + 1/(1+2\lambda_4) - 2\beta(1+\lambda_3, 1+\lambda_4)$$

$$C = 1/(1+3\lambda_3) - 1/(1+3\lambda_4) - 3\beta(1+2\lambda_3, 1+\lambda_4) + 3\beta(1+\lambda_3, 1+2\lambda_4)$$

$$D = 1/(1+4\lambda_3) + 1/(1+4\lambda_4) - 4\beta(1+3\lambda_3, 1+\lambda_4) - 4\beta(1+\lambda_3, 1+3\lambda_4) + 6\beta(1+2\lambda_3, 1+2\lambda_4)$$

$$\beta(u,v) = \text{The beta function} = \int_0^1 t^{u-1}(1-t)^{u-1}\,dt \quad \text{[Press 1992, 215]}$$

The parameters λ_1 and λ_2 do not affect the values of α_3 and α_4. Thus, matching the third and fourth standard moments of a distribution with the Lambda distribution requires solving only two simultaneous equations (both functions of λ_3 and λ_4). The values of λ_1 and λ_2 may then be solved by using the mean and standard deviation of the desired distribution.

One difficulty of solving for the λ_3 and λ_4 parameters is that often more than one combination of Lambda parameters can yield the same first four moments (the higher moments may be different). Thus, Ramberg constrains $\lambda_1\lambda_2$ to be greater 0.0 to narrow down the choices. Tables of common values of standardized skewness and kurtosis for values of λ_3 and λ_4 can be used, or the non-linear equations can be solved simultaneously.

7.2.2.1. Tables

The tables for matching standardized skewness and kurtosis utilize the combinations of λ_3 and λ_4 that result in common distribution shapes (bell-type curves). These common λ_3 and λ_4 values are less than 1.0. Additionally, the absolute value of λ_4 should be greater than the absolute value of λ_3. The tables [Ramberg 1979, 210–214] contain only the

positive values of skewness. For negative values of skewness, λ_3 and λ_4 are switched and the sign of λ_1 reversed.

7.2.2.2. System of non-linear equations

The tables mentioned above were created by solving the non-linear equations through minimizing the error in two functions (skewness and kurtosis). The minimization was accomplished through using the Simplex algorithm [Olsson 1975]. Eq. 7-1 below is the minimization function.

$$Error = (\alpha_3 - \alpha_3(\lambda_3, \lambda_4))^2 + (\alpha_4 - \alpha_4(\lambda_3, \lambda_4))^2 \qquad \textbf{Eq. 7-1}$$

Attempts at applying this function minimization failed due to local minimums and a poor understanding of the characteristics of the Lambda distribution.

7.2.3. Method Followed

The fist attempt at matching the moments for the Lambda distribution through the minimization function involved using a method of steepest descent. A penalty function was implemented to keep λ_3 and λ_4 less than 1.0, and to keep λ_3 and λ_4 the same sign. There seemed to be local minimums, which impeded solving the system. Next a simulated annealing method was used with similar results.

7.2.3.1. Understanding the Minimization Function

After failing to consistently minimize the error function and not finding universally good starting points for λ_3 and λ_4, the minimization function was analyzed. Figure 7-1 below shows the minimization function versus λ_3 and λ_4 for matching α_3 = 1.0 and the α_4 = 5.0.

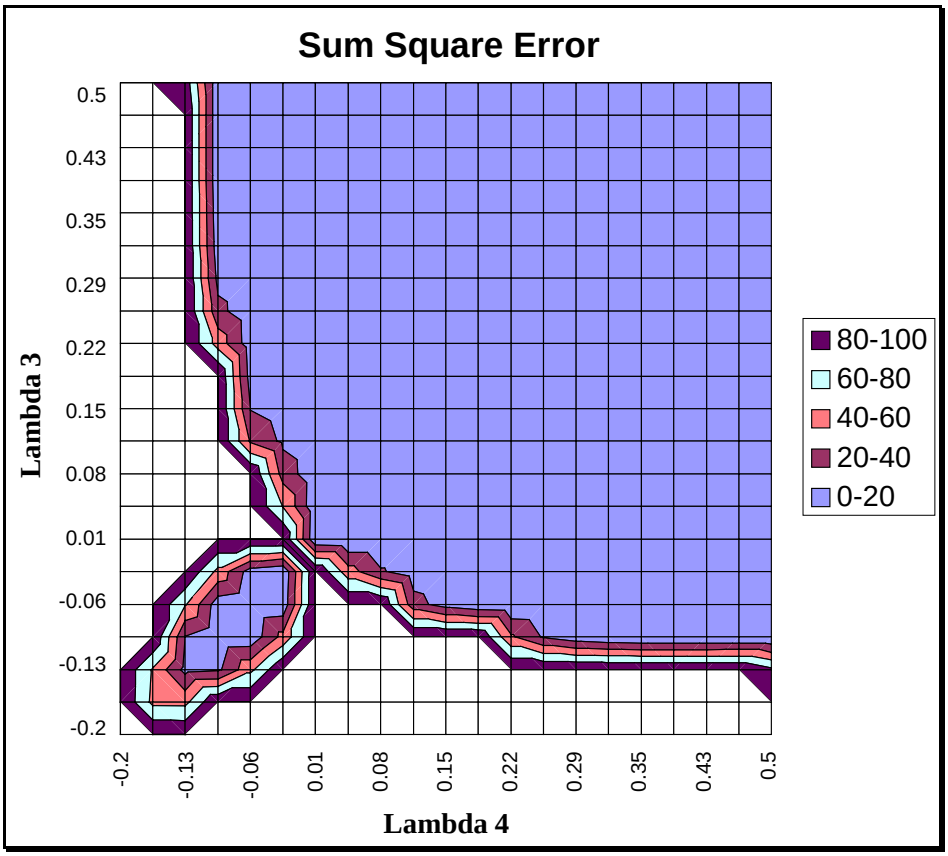


Figure 7-1: Function to Minimize

The surface shown in the figure above represents the minimization function for a desired third and fourth moments of 1.0 and 5.0 respectively. The surface is fairly symmetrical about the line $\lambda_3 = \lambda_4$. Two major areas of possible solutions (error = 0.0) are where λ_3 and λ_4 are less than zero, and when they are greater than zero. There is a division between them that includes the origin.

The two major regions for minimizing the function in the figure above are quadrant one and quadrant three. The “hole” in quadrant three is quite isolated, and would have to be searched separately from the main region in quadrant one. The above minimization function was investigated in great depth to find the areas of local minimums. Figure 7-2 below shows only the areas where the minimization function is less than 2.0.

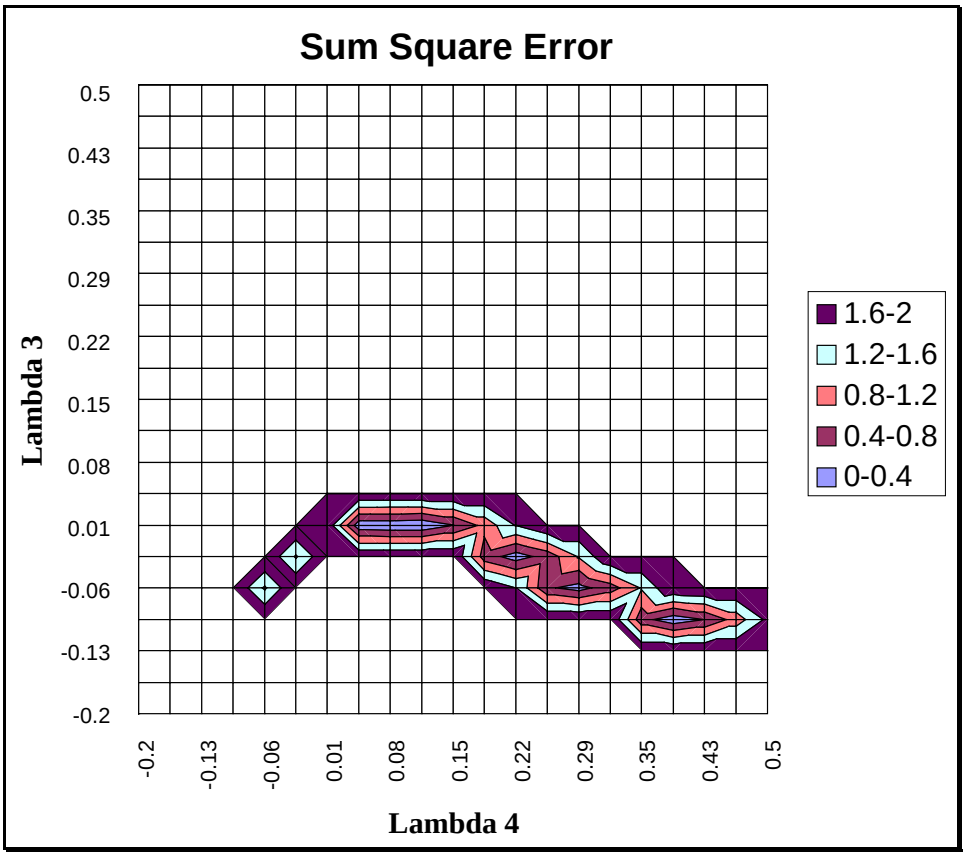


Figure 7-2: Local Minimums for Function

This figure represents the same surface in Figure 7-1, except that only the areas where the function is less that 2.0 are shown. With a finer scale, the local minimums can now be seen. There are six separate local minimums where a minimization algorithm could get stuck. Figure 7-2 above shows that there are six different local minimums for this minimization function. These local minimums are the main reason that function minimization did not result in consistent solutions.

7.2.3.2. Investigation of the Lambda Function

To investigate a better solution to matching the Lambda parameters, the moments of the Lambda distribution were analyzed. Figure 7-1 below shows the surface that represents the skewness of the Lambda distribution versus the parameters λ_3 and λ_4.

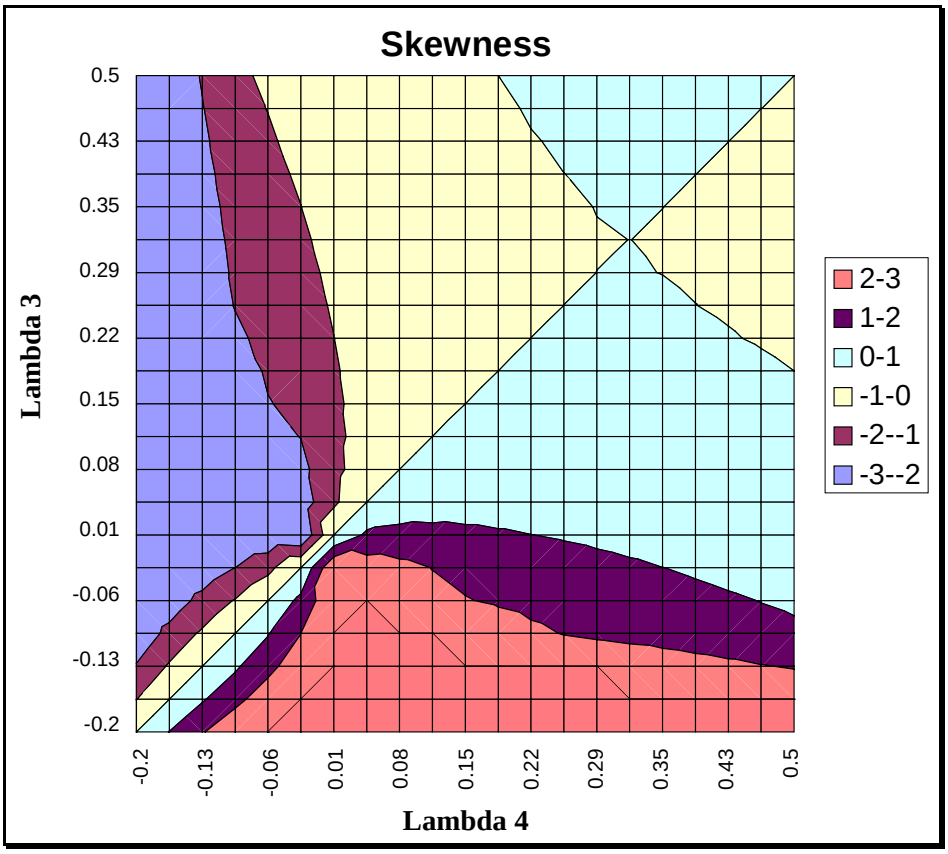


Figure 7-1: Skewness vs. λ_3 and λ_4

The skewness of the Lambda distribution as a function of λ_3 and λ_4 is symmetrical about the line $\lambda_3 = \lambda_4$ except that the sign of the skewness is opposite. The surface is quite steep around the origin. Many of the λ_3 and λ_4 parameters in the standard tables fall into the negative skewness range. But, along with those parameters, λ_2 is also negative. A negative λ_2 causes the skewness to change sign, so the negative skewness range can also be used to match a positive skewness moment.

Figure 7-2 below shows the kurtosis of the Lambda distribution. It is very similar to the minimization function shown earlier. Because λ_2 can change the sign of the skewness, both the skewness and the kurtosis are symmetrical about the line $\lambda_3 = \lambda_4$.

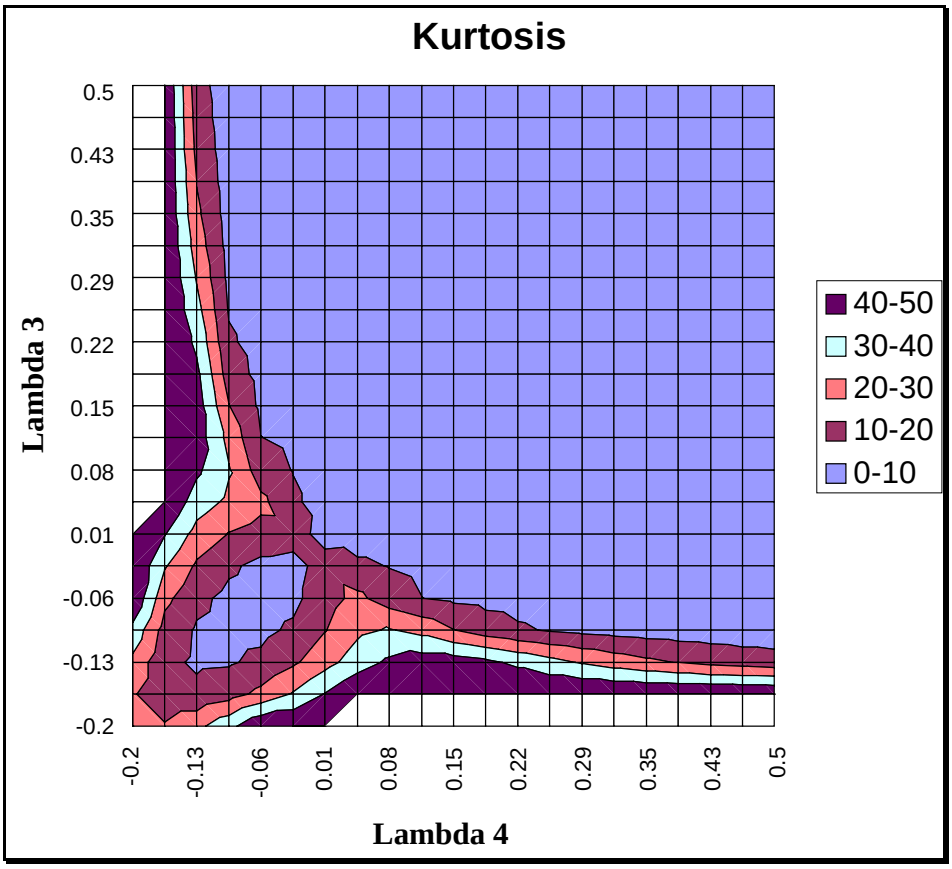


Figure 7-2: Kurtosis vs. λ_3 and λ_4

The kurtosis of the Lambda distribution is also symmetrical about the line $\lambda_3 = \lambda_4$. This surface looks very similar to the minimization function in Figure 7-1. Because the values of kurtosis are normally several times greater than the values of skewness, errors in kurtosis dominate the shape of the minimization function.

7.2.3.3. Successful Algorithm

A solution to matching the moments of the Lambda distribution was determined by plotting the constant skewness and kurtosis curves for the desired moments and finding the intersection point. Figure 7-1 below shows the curves for skewness = 1.0 and kurtosis = 5.0.

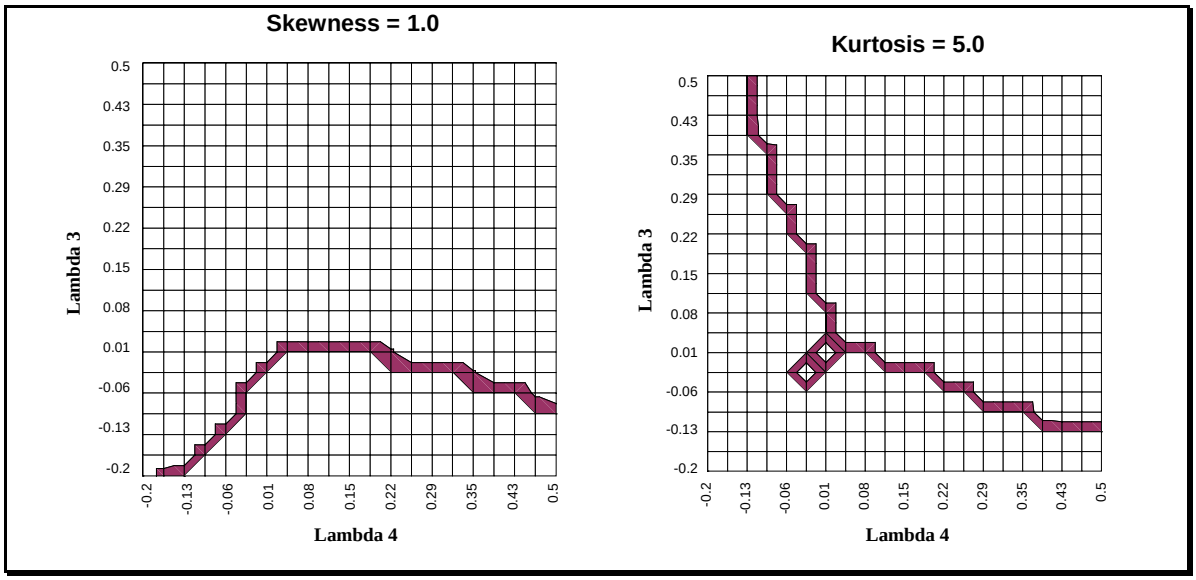


Figure 7-1: Curves for Skewness = 1.0 and Kurtosis = 5.0

The curve representing skewness = 1.0 is a relatively smooth, well-behaved curve. The curve representing kurtosis = 5.0 is not quite as well behaved near the origin. These curves are typical for many of the values of skewness and kurtosis that the Lambda distribution can assume. Because of the well-behaved nature of the skewness curve, it was chosen to be the curve to follow. The algorithm consisted of the following steps.

1. Start at the point $\lambda_3 = \lambda_4 = -0.1$ (any starting point besides the origin will due)
2. Find the gradient of the skewness surface
3. Move in the direction of the gradient to increase the skewness to the desired level (the opposite direction to decrease the level)
4. Follow the skewness curve by going in the direction perpendicular to the gradient
5. Monitor the value of the kurtosis as the skewness curve is traveled
6. When the algorithm strays from the true curve of the desired skewness correct the position as in steps 2 and 3

7. If the value of the kurtosis is crossed, reverse direction and decrease the step size

8. If the value of the skewness and kurtosis are both within the desired accuracy, quit

The above algorithm rapidly and consistently finds the solution to λ_3 and λ_4 that yield the desired moments. Then, if the magnitude of λ_3 is greater than that of λ_4, the values are exchanged and the sign of λ_2 is reversed. Figure 7-2 below shows a sample output of this searching algorithm.

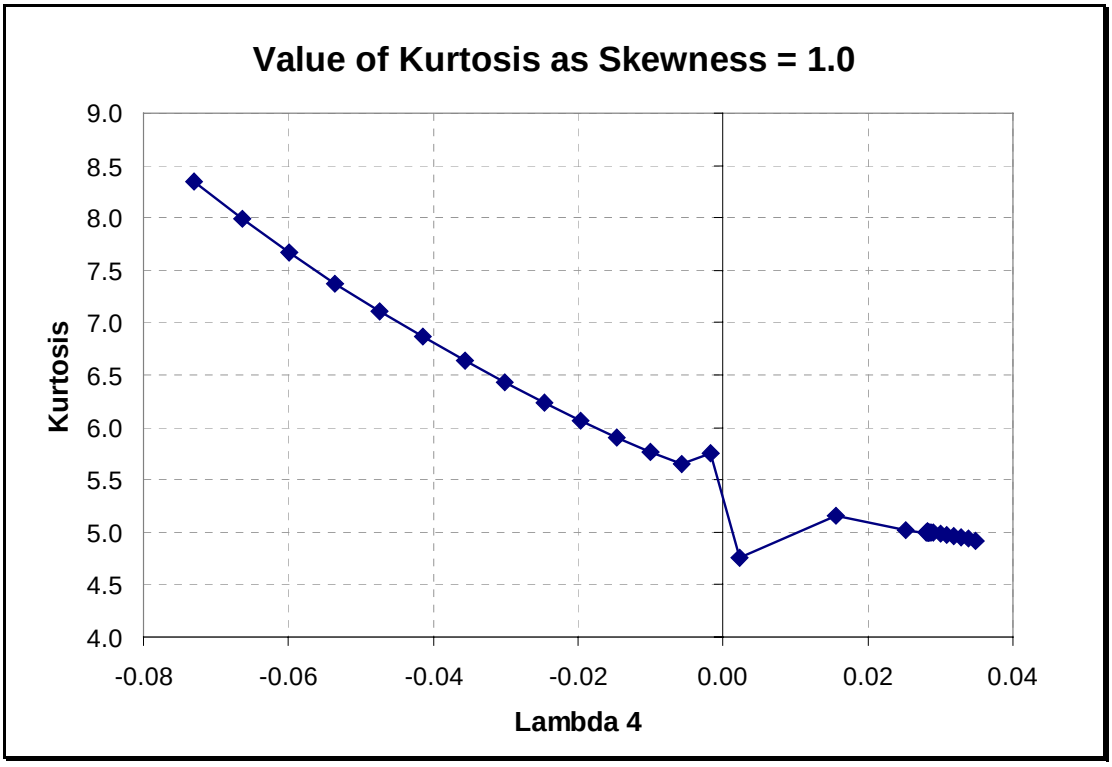


Figure 7-2: Value of Kurtosis while Following the Skewness = 1.0 Curve

The curve above shows a sample of how the algorithm keeps track of the kurtosis as the skewness is kept constant. Around the origin ($\lambda_4 = 0.0$), the skewness curve changes rapidly, causing a jump in kurtosis. When the desired value of kurtosis is passed, the direction is reversed and the step size decreased until the desired accuracy is attained.

Comparing the Lambda parameters obtained through the algorithm outlined above with the parameters in published tables [Ramberg 1979, 210–214] supports the algorithm as being valid. The algorithm must also incorporate some error checking to ensure that valid moments are entered, which are in the range of the Lambda, as shown in Figure 7-1.

Section 7.3. Summary for the Lambda Distribution

This chapter not only presented the state-of-the-art in terms of using the Lambda distribution, but it also demonstrated a new algorithm to solve for the Lambda parameters. The Lambda distribution is a valuable tool in variation analysis to estimate the probability density function for the output distribution. The probability density function could then be used to determine PPM rejects, non-symmetrical quality loss, the mode of the distribution, etc. The Lambda distribution will be used throughout this thesis in estimating PPM rejects from distribution moments.

The Lambda distribution is not always a perfect fit (as shown with the normal distribution), but then again neither is the input information for most analysis problems (issue discussed in Section 13.1).

Chapter 8. Determining the Effect of Errors in Moment Estimates

It has been shown that the method of system moments and Monte Carlo simulation can be used for variation analysis of assemblies with non-normal distributions. The non-normal distributions for each input variable are defined in terms of the first four or eight moments of area about the mean. The method of system moments uses the first eight input moments directly to compute the four resulting moments of the assembly objective. Monte Carlo fits a distribution to the moments for each of the inputs and uses the distributions to generate random assembly data from which the assembly moments are calculated or the PPM rejects counted. Both methods can fit a distribution to the output moments to estimate PPM rejects.

In Section 3.6 it was shown that the accuracy of the mean and variance estimated for a sample of a population is dependent on the sample size. It will be shown in Section 9.2 that the accuracy of the other moments is likewise dependent on sample size. The accuracy of both Monte Carlo and the method of system moments is dependent on the accuracy of the input variable moments.

In this chapter, the effect of errors in the input variable moments on the output distribution moments will first be investigated. Then the effect of errors in the output distribution moments on estimating PPM rejects and quality loss will be estimated. The result will be a better understanding of how errors in the input information and analysis can cause uncertainty in the analysis results.

In order to study the effect of moment errors, a standard measure of error is needed. Percent error is often used for error analysis, but this was rejected for studying moment errors. Since the magnitude of the moments is often near zero, particularly for skewness, the percent error can be quite large. This can give a distorted view of the error.

A better approach is to non-dimensionalize the error in terms of the standard deviation—then each error is sized relative to its own variation. This is similar to the practice of converting a raw normal distribution to a standard normal, having a mean of 0.0, standard deviation of 1.0, skewness of 0.0, and kurtosis 3.0. In fact the standardized values of skewness and kurtosis are also calculated by non-dimensionalizing with respect to the standard deviation as shown below.

$$\alpha_3 = \mu_3 / \sigma^3 \text{ (standardized skewness)} \qquad \textbf{Eq. 8-1}$$

$$\alpha_4 = \mu_4 / \sigma^4 \text{ (standardized kurtosis)} \qquad \textbf{Eq. 8-2}$$

One of the major contributions of this thesis is the new metrics to help estimate accuracy. The standard moment error presented in this chapter is one of the new metrics.

Section 8.1. Defining Standard Moment Errors

A dimensionless standardized error for each moment (contribution of this thesis) is defined as follows:

$SER1 = \dfrac{(\hat{\mu}_1 - \mu_1)}{\sigma}$ Standard error of $\hat{\mu}_1$ (the estimate of μ_1)

$SER2 = \dfrac{(\hat{\mu}_2 - \mu_2)}{\sigma^2}$ Standard error of $\hat{\mu}_2$ (the estimate of μ_2)

$SER3 = \dfrac{(\hat{\mu}_3 - \mu_3)}{\sigma^3}$ Standard error of $\hat{\mu}_3$ (the estimate of μ_2)

$SER4 = \dfrac{(\hat{\mu}_4 - \mu_4)}{\sigma^4}$ Standard error of $\hat{\mu}_4$ (the estimate of μ_4)

$SERi = \dfrac{(\hat{\mu}_i - \mu_i)}{\sigma^i}$ Standard error of $\hat{\mu}_i$ (the estimate of the i^{th} moment)

Where: μ_i is the true value of the i^{th} moment

$\hat{\mu}_i$ is the estimate of the i^{th} moment

$\sigma = \sqrt{\mu_2}$, or the true standard deviation of the distribution

Note: The benchmark Monte Carlo estimates (at one billion samples) of the moments will be taken as the true values of the moments for this thesis.

The main reason for defining the standard error for an estimate of a moment as shown above is to make the error measure more uniform across the spectrum of possible distributions. For example, if a distribution had a very small skewness, an error in estimating the skewness would yield a large relative percent error. Figure 3-2, presented in Chapter 3, shows that the percent change in skewness for the clutch contact angle during a Monte Carlo simulation was much greater than the percent changes in any of the other moments. This is because the actual value of skewness is close to zero.

The other reason for defining the sensitivities, as shown above, is that it follows the same style as the calculation for mean standard error, and similar error for estimates of variance (explained in Section 3.6 for estimating the error in moments for a Monte Carlo simulation). Defining the standard moment error as shown above, rather than dividing the estimate of the moment by the estimate of the standard deviation, and subtracting the standardized true moment, uncouples the errors in moments. If coupled, the error in standard deviation would affect the errors in all of the other moments.

Section 8.2. Defining the One-Sigma Error Bounds

Section 3.6 presented a method of estimating the error for the mean in terms of sample size. The one-sigma bound on the error, or $\sigma_{\mu 1}$, defines the uncertainty in the estimate of the mean. Using this one-sigma error bound for the mean can determine an interval for the true value of the mean, and a confidence level for that interval. Similarly, the one-sigma bounds for the standard moment errors and other objective functions are defined below:

$\sigma_{\mu 1}$	One-sigma bound on the mean
σ_{SER1}	One-sigma bound on the standard moment error for the mean or SER1
σ_{SERi}	One-sigma bound on the i^{th} standard moment error (SER*i*)

σ_{PPM} One-sigma bound for a general objective (PPM rejects is used as an example)

$\sigma_{\%PPM}$ One-sigma bound for a general objective divided by the objective (one-sigma bound on percent error)

These definitions for the one-sigma bounds on error are not unique to this thesis, but are presented here to work together with the definitions of standard moment error. The reason for defining the one-sigma bounds is to deal with uncertainty. Knowing the one-sigma bound for a value can help define (with a certain level of certainty) an interval for which the true value may be. The will play an important part of Chapter 13 where the error in input information is matched with the error in the analysis.

Understanding the one-sigma bound on error is most easily illustrated by examining the one-sigma bound on the mean (introduced in Section 3.6). The one-sigma bound on the mean is equal to the standard deviation of the value being observed divided by the square of the sample size. Figure 8-1 below shows the probability density function (distribution) for estimating the mean of the standard normal distribution versus the sample size.

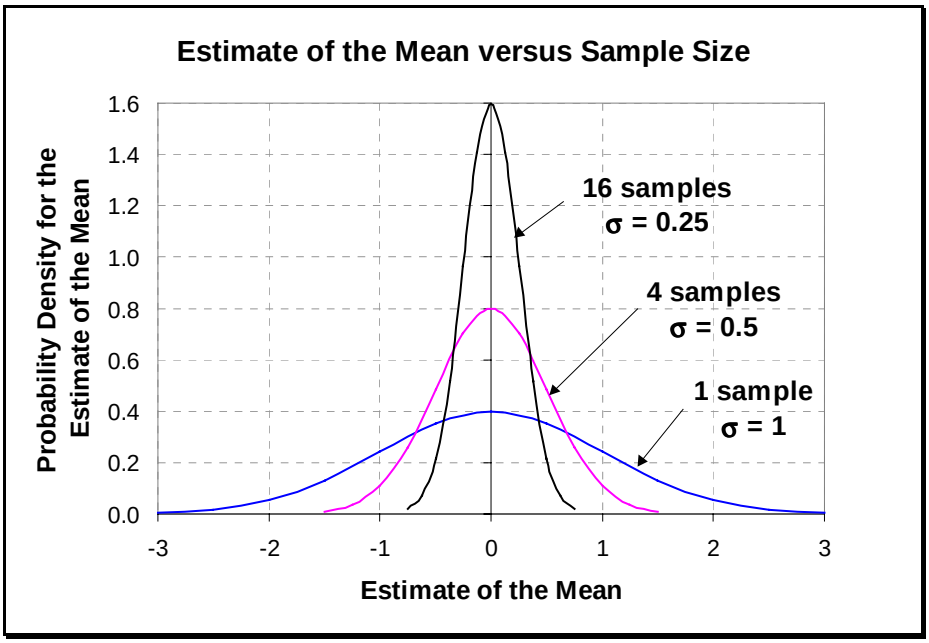


Figure 8-1: Estimating the Mean of the Standard Normal Distribution versus Sample Size

For a sample size of only one, the probability density function for the estimate of the mean is the same as the original standard normal distribution (mean of zero, and standard deviation of one). That means that if only one sample is used to estimate the mean of a value, the variability of that estimate is the same as the variability of the value itself. As the sample size increases to four, the one-sigma bound on the mean is 50 percent of the original. And at 16 samples, the one-sigma bound on the mean in 25 percent of the original.

Section 8.3. Estimating the Effect of Input Variable Moment Uncertainty

The task of defining the input variable distributions is required for all of the variation analysis methods to some degree. For the method of system moments, the first eight moments for each input variable are required. Monte Carlo requires the distributions to be known (that requires knowing or assuming all of the moments for the input variables). The input variable moments can be obtained through assuming an exact distribution,

measuring actual parts being produced, or measuring output from specific processes. Typically, manufacturing data are limited to no more than the first four moments.

Assuming an exact distribution often means assuming that the input variables are normally distributed. This assumption is often valid, as the normal distribution can model phenomena caused by multiple, random, independent events. Differences in machines, operators, materials, environmental conditions, and the many undefined random factors all affect part characteristics. But other distributions can also be chosen. The next most common distribution is the uniform.

Measuring the actual parts or other output from a similar process can also help define input variable distributions. Sometimes a part distribution can even be truncated because inspection is eliminating the tails. The confidence levels for measuring the input variable moments are similar to using Monte Carlo to determine the output distribution moments (see Section 3.6 and Section 9.2). Many samples may need to be taken to get a good estimate of the moments.

If too few samples are used to estimate the input variable distributions, or if the distributions are assumed, the input error should be evaluated, as it will be a limit to the accuracy of any method of analysis applied to the problem. Because the effects of errors in the estimates of the input moments are so dependent on the assembly function and the situation, a general form for the error will not be covered in this thesis.

Figure 8-1 below shows the magnitude of error that might be expected for the first eight moments of an input variable, if the input distribution is estimated from 1,000 samples. The standard deviation of the standard moment errors from ten runs (each of 1,000 samples) is used as the estimate for the magnitude of the one-sigma bound on the error.

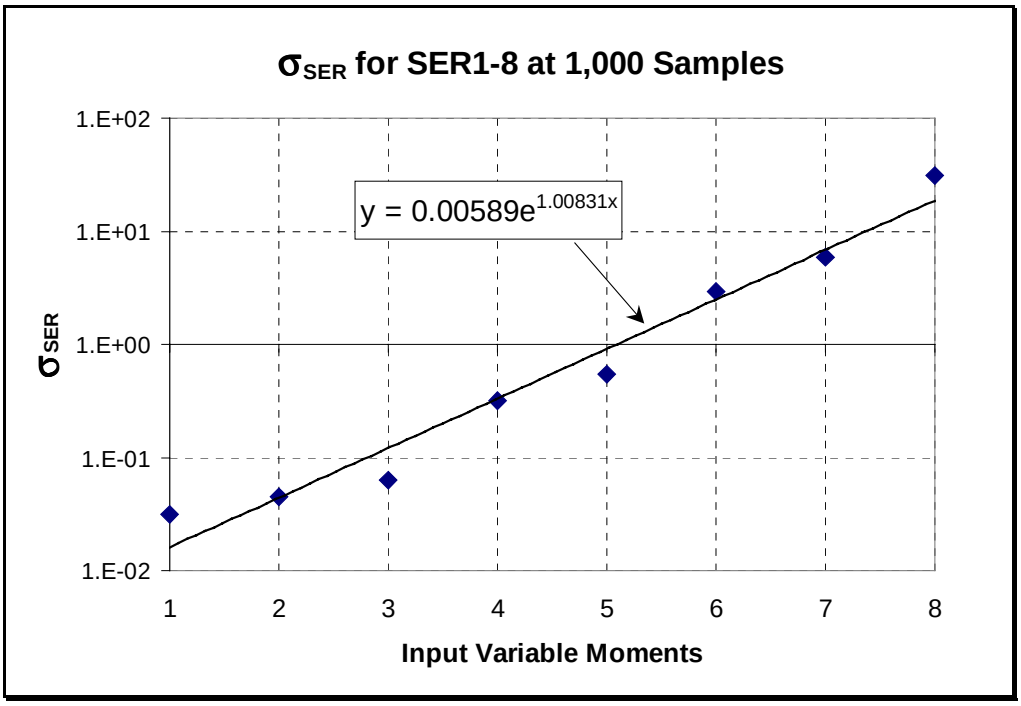


Figure 8-1: One-Sigma Error Bound for SER1–8 at 1,000 Samples

Ten different runs, consisting of 1,000 samples each, of a standard normal distribution, were used to estimate the one-sigma bound for the standard moment errors for the first eight moments of a hypothetical input variable distribution. The equation shown is of the trend line that best fits the data. Figure 8-1 above shows that the errors in the moments increase exponentially for a fixed sample size (the plot is on log scale). Only the magnitude of the errors is needed at this point. The errors for the first four moments versus sample size will be determined with more accuracy in Section 9.2. But how sensitive is the first four moments (using MSM) to errors in the first eight moments of the input variables?

The first eight moments for the input distribution for the hub radius in the clutch assembly were varied one at a time to determine their effect on the output distribution. The method of system moments was performed on each set of moments to determine the output moments. The method of system moments is the natural choice to investigate the

effects of input variable moments, as it uses them directly to calculate the four output distribution moments.

Table 8-1 below shows the sensitivities of the output moments to the standard error for the input moments of the hub radius (refer to Figure 2-2 for a drawing of the clutch assembly). The sensitivities were calculated by dividing the standard moment error of the output moment by the standard moment of the input distribution (from the normal case). The hub radius was chosen because it contributed over 81 percent of the total variation of the contact angle (see Table 6-2).

Table 8-1: Sensitivity of Output Moments to Input Moments

		Output Moment Error			
		SER1	**SER2**	**SER3**	**SER4**
Input Moment Error	**SER1**	**0.9063**	0	0	0
	SER2	0.0127	**0.8188**	0.0379	**0.8827**
	SER3	0	0.0230	**0.7411**	0.0376
	SER4	0	1.6E-04	0.0313	**0.6715**
	SER5	0	0	4.4E-04	0.0378
	SER6	0	0	2.1E-06	8.0E-04
	SER7	0	0	0	7.5E-06
	SER8	0	0	0	2.6E-08

The sensitivities are in terms of the standard moment error (SER) of the output distribution (contact angle) that is caused by 1.0 SER in the input distribution for the hub radius (a). The bolded values are the largest sensitivities.

Now that the sensitivities for the output distribution moments are estimated in Table 8-1, they can be combined with the magnitudes of the standard moment errors for the input variables using 1,000 samples to estimate the input variable distributions (see Figure 8-1). The magnitude of the errors in output moments caused by errors in each of the first eight input variable distribution moments is thus estimated in Table 8-2 below.

Table 8-2: Expected Error in Moments from Errors of an Input Distribution Estimated from only 1,000 Samples

		Output Moment Error				
		Delta	**SER1**	**SER2**	**SER3**	**SER4**
Input Moment Error	**SER1**	0.032	**0.0286**	0	0	0
	SER2	0.045	5.7E-04	**0.0366**	0.0016	0.0395
	SER3	0.063	0	0.0015	**0.0469**	0.0024
	SER4	0.317	0	5.1E-05	0.0099	**0.2130**
	SER5	0.911	0	0	4.0E-04	0.0344
	SER6	2.496	0	0	5.2E-06	0.0020
	SER7	6.843	0	0	0	5.1E-05
	SER8	18.76	0	0	0	4.9E-07

The delta values for the first four input distribution moments of the hub radius are from Eq. 9-1 through Eq. 9-4, presented later. The delta values for the fifth through eighth input distribution moments are estimated form the empirical equation shown in Figure 8-1. The greatest output standard moment error is for the kurtosis caused by the error of kurtosis of the input variables.

For each of the output moments in the table above, the major contributor of error is the error in that same moment estimate for the input variables. The errors from input moments five through eight have little effect on output distribution moments one through four. Because the sensitivities of each of the output distribution moments with respect to the same input distribution moments is close to one, the standard moment error of the output moments is the same order of magnitude as the input distribution moment errors.

For the clutch assembly problem the largest standard moment error for the output distribution moments is for the kurtosis, and it is caused by a poor estimate of the kurtosis of the input variable distributions. In this case the error in kurtosis for the output distribution is about five times the error in the other moments. Later in this chapter it will be shown (see Figure 8-1) that the sensitivity of PPM rejects is only two or three times greater for SER2 than for SER4. Thus the majority of the error in estimating the PPM

rejects for an assembly could easily come from a poor estimate in the kurtosis of the input variables.

The following sections will determine the sensitivities several output functions (PPM rejects, quality loss, mode of distribution, etc.) with respect to the four output distribution moments. The results from this section and the following sections will help to understand how uncertainty in both input information and analysis can affect the certainty of the results.

Section 8.4. Explicitly Determining the Sensitivities for Quality Loss

The sensitivities of some objective functions with respect to the standard error in moments can be determined explicitly. This section will determine the sensitivities of the Taguchi quality loss function explicitly. The next section will determine the sensitivities for several other objective functions by performing an error analysis (or linear variation analysis) procedure.

The quality loss function was shown by Eq. 6-5 to be only a function of the mean and variance of the assembly function. Eq. 8-1 below shows the general form for the quality loss where m is the value for the assembly function at the minimum quality loss and K is the cost constant. The estimates of the mean and variance are used in place of the true values to determine the effect of errors.

$$L = K\hat{\mu}_2 + K\left(\hat{\mu}_1 - m\right)^2 \qquad \textbf{Eq. 8-1}$$

To calculate the sensitivities of the Taguchi quality loss function, the derivatives of the loss function are taken with respect to SER1 and SER2. These derivatives are shown below in Eq. 8-2 and Eq. 8-3, along with the values of the sensitivities at the nominal values of the moments.

$$\frac{\partial L}{\partial SER1} = 2K(\hat{\mu}_1 - m)\frac{\partial \hat{\mu}_1}{\partial SER1} = 2K(\hat{\mu}_1 - m)\sigma \qquad \textbf{Eq. 8-2}$$

$$\frac{\partial L}{\partial SER1} = 2(55.56)(7.0149 - 7.0184)(0.2197) = \$ - 0.0854$$

$$\frac{\partial L}{\partial SER2} = K\frac{\partial \hat{\mu}_2}{\partial SER1} = K\sigma^2 \qquad \textbf{Eq. 8-3}$$

$$\frac{\partial L}{\partial SER2} = 55.56(0.2197)^2 = \$2.6818$$

The sensitivity of the quality loss with respect to SER1 is a function of the distance of the mean from the minimum loss point. For this case, the mean is very close to the minimum cost point, and the sensitivity is very low. If the mean were 0.5σ from the minimum cost point the magnitudes of the sensitivities would be the same with respect to SER1 and SER2.

Now that the sensitivities are known the one-sigma bound on the error for the quality loss can be estimated using a linear variation analysis approach as shown below in Eq. 8-4.

$$\sigma_L = \sqrt{(S_{SER1}\sigma_{SER1})^2 + (S_{SER2}\sigma_{SER2})^2} \qquad \textbf{Eq. 8-4}$$

Where: σ_{SERi} = The sensitivity of quality loss to the i^{th} standard moment error

The rest of this chapter will be devoted to investigating the sensitivities of the remaining objective functions. A two-level DOE will be performed to estimate the sensitivities, as the partial derivatives are not easily obtained explicitly.

Section 8.5. Two-Level DOE to Determine the Effect of Errors in the Moments

A two-level experimental design was chosen to find the sensitivities of the objective functions to errors in moments. The problem will be treated very similarly to a variation analysis problem where the input variables are the moments of an output distribution, and the response variables are the objective functions (rejects, quality loss, etc.) The Lambda

distribution will be used where necessary to calculate the objective function from the output distribution moments.

A full two-level design will be used because there are only four input variables, and only linear sensitivities are desired. The sensitivities of the objective functions to errors in the moments will be performed for nominal output distribution moments for the clutch contact angle. The benchmark values from the one billion sample Monte Carlo analysis will be taken as the exact values for the output distribution moments. The sensitivities found will be presented as reasonable ballpark estimates for most variation analysis problems.

8.5.1. Design Setup

The levels for the moments were chosen to be plus and minus 0.005 standard error (SER). Because the design is a two-level design, the levels are to be at plus and minus one sigma of the input variables. In this case the one-sigma bounds on the error for the standard moment errors (σ_{SER}) will be used as the change from nominal for the levels ($\Delta\mu_k$). The calculation of the levels, $\Delta\mu_k$, for the moments is quite straightforward as shown in Eq. 8-1 and Eq. 8-2 below.

$$SER\,k = \frac{\Delta\mu_k}{\sigma^k} \qquad \textbf{Eq. 8-1}$$

$$\Delta\mu_k = \sigma^k SER\,k \qquad \textbf{Eq. 8-2}$$

Table 8-1 below shows the calculations of the $\Delta\mu_k$ for each of the moments. The higher moments are changed much less than the mean and variance due to the standard moment errors being the same.

Table 8-1: Determining Levels for the Moments

Variable	Nominal	$\Delta\mu_k$
μ_1	7.01495	1.10E-03
μ_2	0.04825	2.41E-04
μ_3	-0.00100	5.30E-05
μ_4	0.00704	1.16E-05

The $\Delta\mu_k$, or σ_{SER}, for each of the levels is 0.005. This σ_{SER} level is not unreasonable for the mean and variance for a typical Monte Carlo simulation of about one hundred thousand samples, see Chapter 9. The setup of the design (levels and values of the moments and values) is shown below in Table 8-2. There are a total of 16 runs. Because the design is a full 2^4 factorial, the interaction effects could be accurately estimated. The skewness is negative, so the upper-level value of skewness is closer to zero. This will not affect the magnitude of the sensitivity of any of the objective functions with respect to skewness because the sensitivity is based on the change in skewness, not the absolute magnitude of the skewness.

Table 8-2: Two-Level Design for Moments Sensitivity Estimation

Run	μ_1	μ_2	μ_3	μ_4	μ_1	μ_2	μ_3	μ_4
1	-1	-1	-1	-1	7.01385	0.04801	-0.00105	0.00703
2	-1	-1	-1	+1	7.01385	0.04801	-0.00105	0.00705
3	-1	-1	+1	-1	7.01385	0.04801	-0.00095	0.00703
4	-1	-1	+1	+1	7.01385	0.04801	-0.00095	0.00705
5	-1	+1	-1	-1	7.01385	0.04850	-0.00105	0.00703
6	-1	+1	-1	+1	7.01385	0.04850	-0.00105	0.00705
7	-1	+1	+1	-1	7.01385	0.04850	-0.00095	0.00703
8	-1	+1	+1	+1	7.01385	0.04850	-0.00095	0.00705
9	+1	-1	-1	-1	7.01605	0.04801	-0.00105	0.00703
10	+1	-1	-1	+1	7.01605	0.04801	-0.00105	0.00705
11	+1	-1	+1	-1	7.01605	0.04801	-0.00095	0.00703
12	+1	-1	+1	+1	7.01605	0.04801	-0.00095	0.00705
13	+1	+1	-1	-1	7.01605	0.04850	-0.00105	0.00703
14	+1	+1	-1	+1	7.01605	0.04850	-0.00105	0.00705
15	+1	+1	+1	-1	7.01605	0.04850	-0.00095	0.00703
16	+1	+1	+1	+1	7.01605	0.04850	-0.00095	0.00705

Sixteen runs will be performed to estimate the sensitivities of the objective functions to each of the four moments. The levels for the moments are centered around the nominal distribution for the clutch contact angle.

8.5.2. Using the Lambda Distribution to Calculate the Objective Functions

The Lambda distribution will help determine the sensitivities of the objective functions to the moments by modeling the experimental moments, and calculating the values of the objective functions. The Lambda distribution is useful because it can model distributions of a wide variety of moments (see Figure 7-1 in Section 7.1). Once the Lambda distribution parameters are determined for a set of distribution moments, the different objective functions can be easily calculated.

The following are the objective functions to be analyzed by experimental design:

- Lower PPM (parts per million) rejects at about 1.0, 2.0, 3.0, 4.0, and 5.0 standard deviations from the mean
- Upper PPM (parts per million) rejects at about 1.0, 2.0, 3.0, 4.0, and 5.0 standard deviations from the mean
- Total PPM (parts per million) rejects at about 1.0, 2.0, 3.0, 4.0, and 5.0 standard deviations from the mean
- Quality Loss (to be compared to the explicitly calculated sensitivities)
- Location of the mode (point of highest probability or peak) of the distribution

The calculations for only the total rejects at about 3.0 standard deviations will be demonstrated below; the other objective functions are calculated using the same technique. This experiment will help to estimate the effect of errors in the first four moments on the different objective functions.

8.5.3. Calculating the Sensitivity for Total PPM Rejects at 3.0 Standard Deviations

Only the calculations for the sensitivities of the total rejects at 3.0 standard deviations will be shown here. The calculations for the other objective functions are very similar, and will be summarized after. The response of total rejects for each of the 16 runs is shown below in Table 8-1.

Table 8-1: Response of Runs for Total PPM Rejects

Run	P(lower)	P(upper)	PPM Rejects
1	0.00190	0.00099	**2,892.61**
2	0.00194	0.00102	**2,961.51**
3	0.00185	0.00103	**2,885.93**
4	0.00189	0.00106	**2,954.65**
5	0.00173	0.00089	**2,617.31**
6	0.00177	0.00092	**2,687.00**
7	0.00168	0.00093	**2,609.16**
8	0.00173	0.00095	**2,679.92**
9	0.00184	0.00103	**2,869.98**
10	0.00188	0.00106	**2,938.62**
11	0.00179	0.00107	**2,865.86**
12	0.00183	0.00110	**2,933.05**
13	0.00167	0.00093	**2,593.45**
14	0.00171	0.00095	**2,664.16**
15	0.00163	0.00096	**2,588.14**
16	0.00167	0.00099	**2,657.31**

These parts per million rejects were calculated by fitting a Lambda distribution for the moments of each of the runs, and then using the Lambda distribution to calculate the probability of being above and below the limits. Once the objective function is calculated for each of the runs, the contribution to variance, percent contribution, and sensitivity for each of the effects is calculated. The calculation of the sensitivities is shown below in Eq. 8-1.

$$SP_i = \frac{\sigma_{\%PPM,SERi}}{\sigma_{SERi}} = \left(\frac{\sigma_{PPM,SERi}}{PPM} \right) \frac{1}{\sigma_{SERi}} \qquad \textbf{Eq. 8-1}$$

Where:

$\sigma_{\text{PPM, SER}i}$ = The standard deviation in PPM rejects (see Section 8.2) caused by SER*i* (the square root of the contribution to variance)

$\sigma_{\%\text{PPM, SER}i}$ = The standard deviation in percent PPM rejects caused by SER*i* (the square root of the contribution to variance)

$\sigma_{\text{SER}i}$ = The one-sigma bound on the standard moment error for the i^{th} output distribution moment

PPM = The average PPM rejects for the response

The reason for defining the sensitivity in terms of percent error in PPM rejects instead of just the error in PPM rejects is to make to sensitivities more usable for ranges of quality levels. Eq. 8-2 below shows how the one-sigma bound on the percent error in PPM rejects is calculated from the sensitivity percents (SP) for each of the moments.

$$\sigma^2_{\%PPM} = \sum_{i=1}^{4} \left(\sigma_{\%PPM,SERi}\right)^2 \qquad \textbf{Eq. 8-2}$$

$$\sigma^2_{\%PPM} = \sum_{i=1}^{4} \left(SP_i \sigma_{SERi}\right)^2$$

The SP calculations for the PPM rejects at 3.0 sigma are shown in Table 8-2 below. The σ_{SER} for each of the moments in the experiment is the same (0.005). The percent contribution and SP for the interaction effects are also calculated to check the assumption that they can be neglected.

Table 8-2: Summary Table for Calculating Sensitivity of Total Rejects

	PPM Effect for Level			Percent	SP
μ	-1	+1	Variance	Contribution	(Sensitivity)
1	22,288	22,111	123.1	**0.61%**	**0.800**
2	23,302	21,096	19,005.5	**93.45%**	**9.936**
3	22,225	22,174	10.0	**0.05%**	**0.228**
4	21,922	22,476	1,197.9	**5.89%**	**2.495**
1,2	22,201	22,198	0.0	**0.00%**	**2.826**
1,3	22,196	22,203	0.2	**0.00%**	**6.242**
1,4	22,201	22,198	0.0	**0.00%**	**2.142**
2,3	22,201	22,197	0.1	**0.00%**	**3.742**
2,4	22,196	22,203	0.2	**0.00%**	**6.223**
3,4	22,199	22,199	0.0	**0.00%**	**0.000**
error	-	-	0.1	**0.00%**	**-**
Total Variation			**20337.15166**	**100%**	

The PPM effects for each of the levels is just the sum of the runs with that level for that moment. The variance and percent contribution were calculated similarly to the method in Section 6.4. The sensitivity percent (SP) for the interaction terms were calculated by dividing the square root of the contribution to variance by the σ_{SER} for both of the moments in the interaction.

Even though the sensitivities for the interactions are of similar magnitude as the others, they do not contribute significantly to the overall variation. Their contribution to the variation is calculated by multiplying their sensitivity by the σ_{SER} for both of the components and squaring. With the SER for each being on the order of 0.005 the variation due to the interactions becomes very small.

The contribution to variance for the standard moment error of the variance and kurtosis contribute together over 99 percent of the total variation in PPM rejects. The reason that the mean and skewness have little effect on total PPM rejects is because a reduction in rejects for one limit (tail of the distribution) is usually accompanied by an increase of rejects for the other limit.

Section 8.6. Summary of Sensitivity Results

8.6.1. Results for all of the Objective Functions

Instead of presenting a table with the sensitivities and percent contributions of the errors in moments to the rejects, they will be presented in graphical form. The magnitudes of the sensitivities are wanted to get an estimate for accuracy of predicting rejects from the output distribution moments. Figure 8-1 below shows the percent sensitivities for the errors in moments.

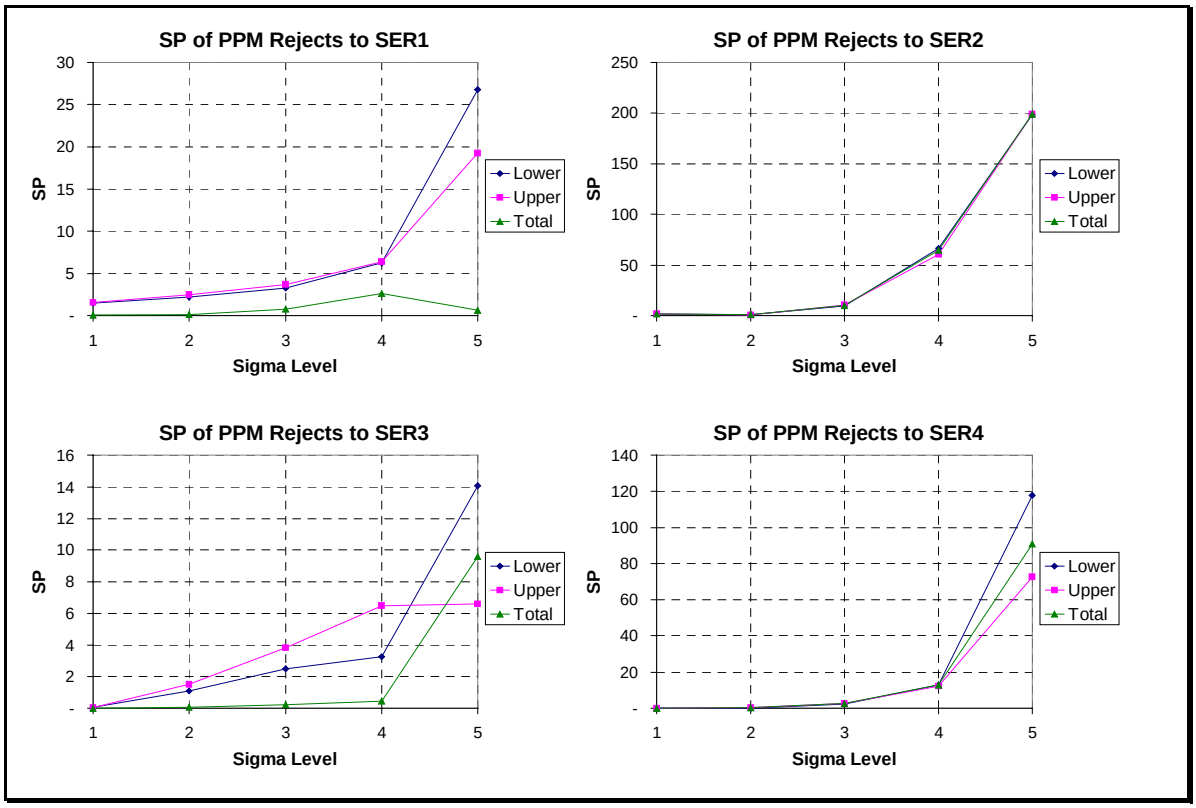


Figure 8-1: Sensitivity Percent of PPM Rejects to Standard Moment Errors

The sensitivities shown above are greater in virtually every case when trying to estimate PPM rejects farther out in the tails of the distribution. The sensitivities with respect to SER1 and SER3 are less for the total rejects than for the lower or upper rejects in most cases. The calculation of the sensitivities did not include the sign of their effect; but on closer observation of the data, the signs for the upper and lower sensitivities are indeed opposite.

In general, the sensitivities for the PPM rejects are greater with respect to SER2 and SER4 than to SER1 and SER3. The axes on the graphs are not scaled to compare the relative magnitudes of the sensitivities between graphs, but within the graphs. On the other hand, the percent contribution charts below, in Figure 8-2 and Figure 8-3, show the relative contribution of the errors moments for total and one-sided rejects.

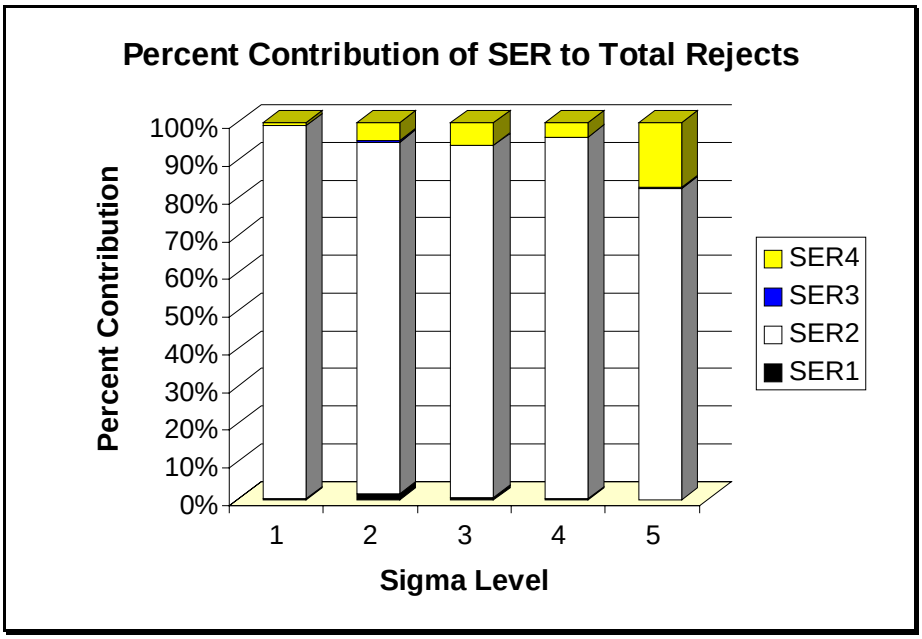


Figure 8-2: Percent Contribution of SER to Total Rejects versus Sigma

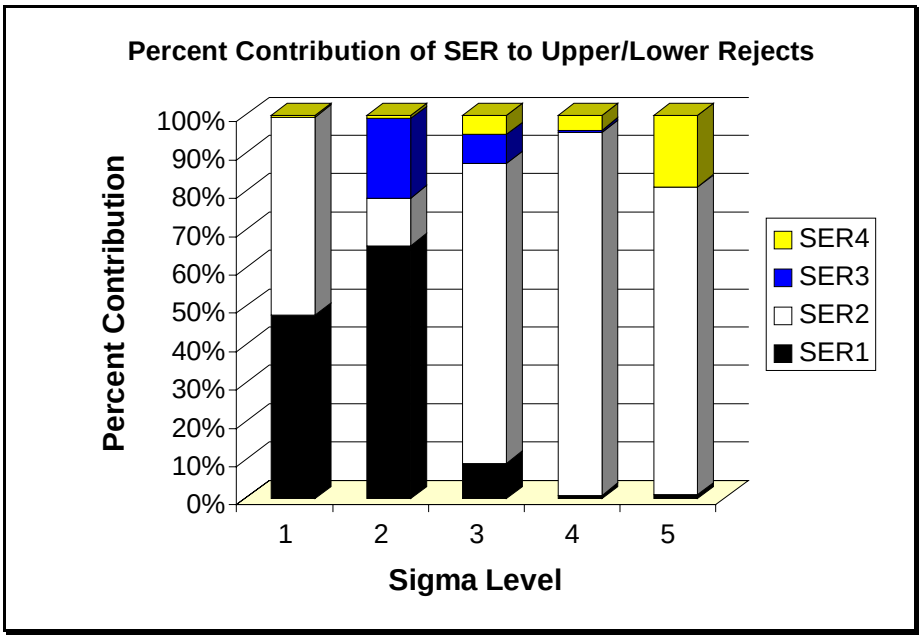


Figure 8-3: Percent Contribution of SER to Upper/Lower Rejects versus Sigma

For total rejects, the error in the mean or skewness has very little effect on the total rejects. The error from variance contributed the most, followed by the error from kurtosis.

The percent contributions for the upper and lower rejects were averaged because they were so similar. The error in variance (SER2) seems to again play the major role in the region most common in variation analysis (three to four sigma). The problem is that the standard moment errors for all of the moments were the same. A more realistic view would be to use standard moment errors according to the accuracy of the analysis method (see Chapter 9).

For both total PPM rejects and lower/upper PPM rejects, the error in SER2 was the greatest contributor in the region of most interest in variation analysis. Table 8-1 below shows the absolute sensitivities (not as a percent of the mean response) and the percent contributions of the standard moment error for the location of the mode and the quality loss. The quality loss is calculated with the Taguchi quality loss function described in Section 6.1.

Table 8-1: Summary Results for the Location of the Mode and Quality Loss

Location of Mode	SER1	SER2	SER3	SER4
Sensitivity	0.21967	0.07946	0.16717	0.01575
%Contribution	58.3%	7.6%	33.8%	0.3%
Quality Loss	SER1	SER2	SER3	SER4
Sensitivity	0.08420	2.68292	0	0
%Contribution	0.1%	99.9%	0%	0%

The location of the mode of a distribution is most sensitive to the mean and then the skewness. The quality loss is most sensitive to the variance and then the mean. The sensitivities of the quality loss function that were calculated by the experimental design are very close to those calculated explicitly (see Eq. 8-2 and Eq. 8-3). This comparison demonstrates the effectiveness of the design of experiments for problems where the response is explicit as well as implicit.

Chapter 9. Estimating the Effectiveness of Monte Carlo Simulation

Chapter 3 presented the state-of-the-art in terms of Monte Carlo simulation for estimating the accuracy. Eq. 3-1 was used to estimate the one-sigma bound on the error of estimating the PPM rejects. Eq. 3-1 was used to estimate the error for the mean, and Eq. 3-2 for the variance. This chapter will investigate the standard moment errors for the first four moments of a distribution as a function of sample size. These results will then be combined with the results on sensitivities from the previous chapter to estimate the error in using the Lambda distribution for calculating PPM rejects. The accuracy of using the moments and Lambda to estimate the PPM rejects will then be compared to counting the PPM rejects.

Section 9.1. Graphically Representing Error versus Sample Size

A graphical demonstration of the error in PPM rejects versus sample size will now be presented. The percent error for estimating the PPM rejects by counting the actual rejects during the simulation (clutch problem presented in Section 3.8), along with the one-sigma bounds on the error, is shown below in Figure 9-1.

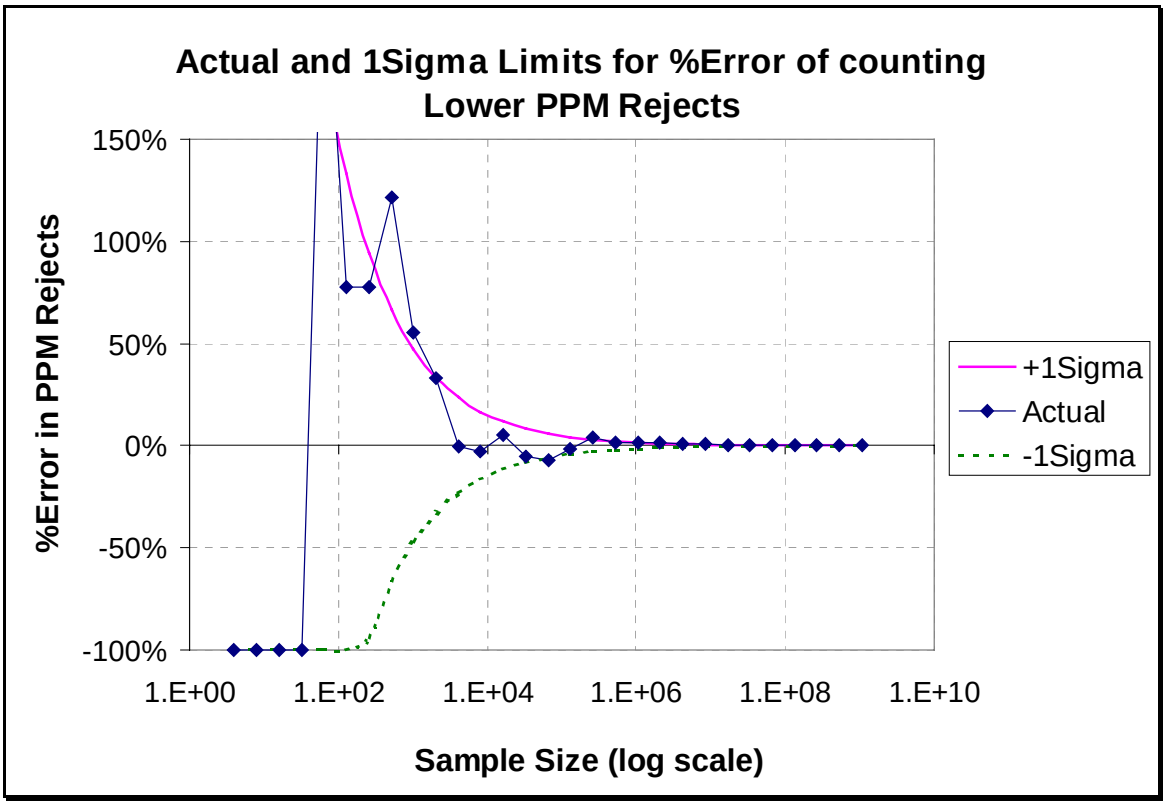


Figure 9-1: Percent Error for Counting Lower Rejects with Monte Carlo Showing One-Sigma Bounds for the Error

The +1 and –1 sigma limits in the figure above show the sixty-eight percent confidence range for the percent error of PPM rejects. The absolute lower bound is –100 percent, as any lower would be predicting negative rejects. The one-sigma bound estimates are calculated directly from Eq. 3-1.

Figure 9-1 illustrates the percent error versus the confidence limits up to about 100,000 samples; afterward the error is too small to be seen on the graph clearly. Therefore, the same information is displayed on a log scale graph below in Figure 9-2. In order to plot the percent error on a log scale, the absolute value of the error was taken. Additionally, the confidence limit bounds are symmetrical, and thus form a single bound. All of the future graphs of error versus sample size will use this same format.

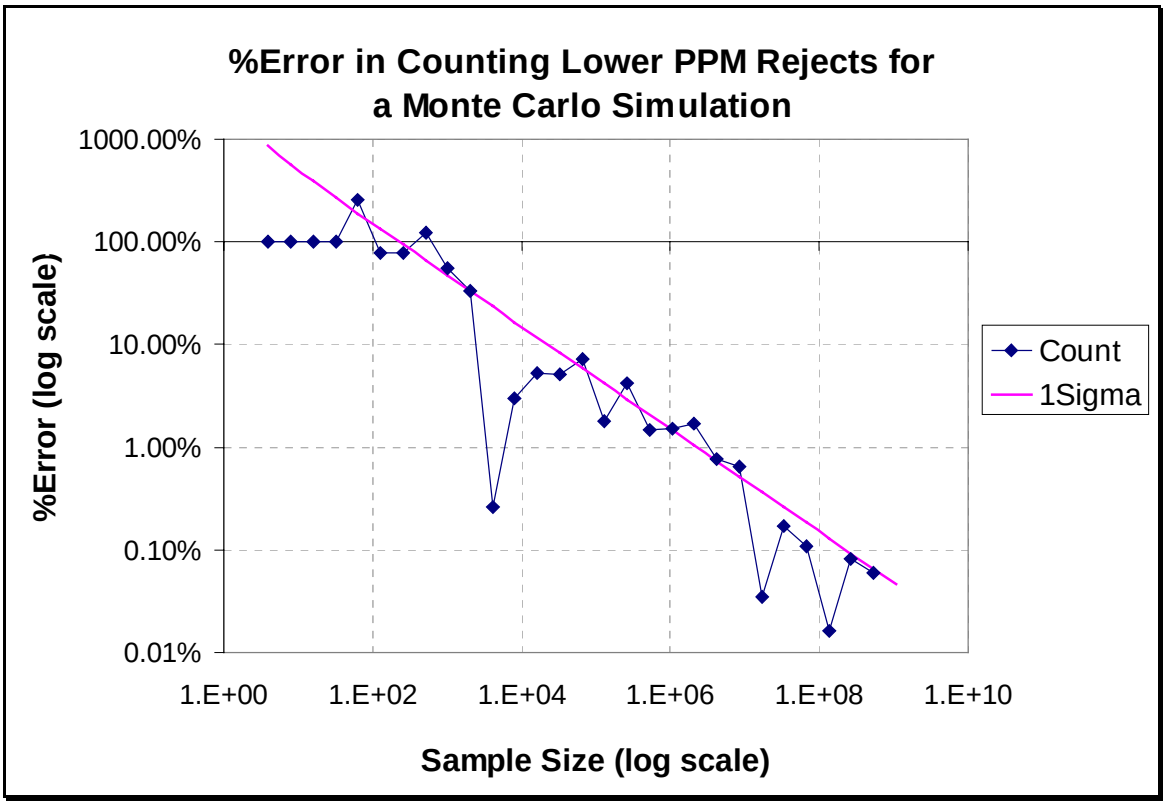


Figure 9-2: Percent Error for Counting Rejects with Monte Carlo versus Sample Size

This graph shows the absolute value of the percent error in order to plot the error on a log scale. This graph allows the errors for all of the samples to be viewed together relative to the one-sigma bound. It does appear that Eq. 3-1 generally describes the error of using Monte Carlo to count rejects. The term "count" refers to the method of counting rejects during the Monte Carlo simulation and then calculating the resulting PPM rejects.

Section 9.2. Error in Estimating Moments with Monte Carlo

The estimate for all of the moments of the output distribution improves as the sample size increases. The accuracy of the estimate of the mean of the output distribution was presented in Chapter 3, now the corresponding level of SER1 (see a definition of the standard errors in Section 8.1) is presented below in Eq. 9-1.

$$\frac{\hat{\mu}_1 - \mu_1}{\sigma/\sqrt{n}} = \text{Standard Normal Distribution} \qquad \textbf{Eq. 9-1}$$

$$\sigma_{SER1}^2 = \text{Variance}\left(\frac{\hat{\mu}_1 - \mu_1}{\sigma}\right) = \frac{1}{n}$$

$$\sigma_{SER1} = \frac{1}{\sqrt{n}}$$

The standard moment error for the mean decreases with the square root of *n* (sample size). Using SER1 instead of just the mean error in the mean itself eliminates having to scale the estimate of error by the standard deviation. The one-sigma error bound for the estimate of the variance is a little more complicated. The distribution for estimating the accuracy of the variance was also presented in Chapter 3, but now the subsequent error level for SER2 is presented below in Eq. 9-2.

$$\frac{(n-1)\hat{\mu}_2}{\mu_2} = \text{Chi - Square Distribution :} \qquad \textbf{Eq. 9-2}$$

$$\text{mean} = (n-1), \text{ variance} = 2(n-1)$$

$$\text{Variance}\left(\frac{(n-1)\hat{\mu}_2}{\mu_2}\right) = 2(n-1)$$

$$\text{Variance}\left(\frac{\hat{\mu}_2}{\mu_2}\right) = \frac{2}{(n-1)}$$

$$\text{Variance}\left(\frac{\hat{\mu}_2}{\mu_2} - \frac{\mu_2}{\mu_2}\right) = \frac{2}{(n-1)}$$

$$\sigma_{SER2} = \sqrt{\frac{2}{(n-1)}}$$

The errors for the skewness and kurtosis are not as easy to calculate as with the mean and variance. Using data from several Monte Carlo type simulations, the equations for the one-sigma error bounds for SER3 and SER4 were found to be approximately the following equations.

$$\sigma_{SER3} = \sqrt{\frac{4}{n-2}} \qquad \textbf{Eq. 9-3}$$

$$\sigma_{SER4} = \sqrt{\frac{100}{n-6}}$$ **Eq. 9-4**

To visualize the probability distribution for SER4 at n = 10,000, figure shows a histogram of SER4 for 1,000 runs (or cycles) of Monte Carlo (each at 10,000 samples). Again the clutch assembly is the assembly function for the Monte Carlo simulation.

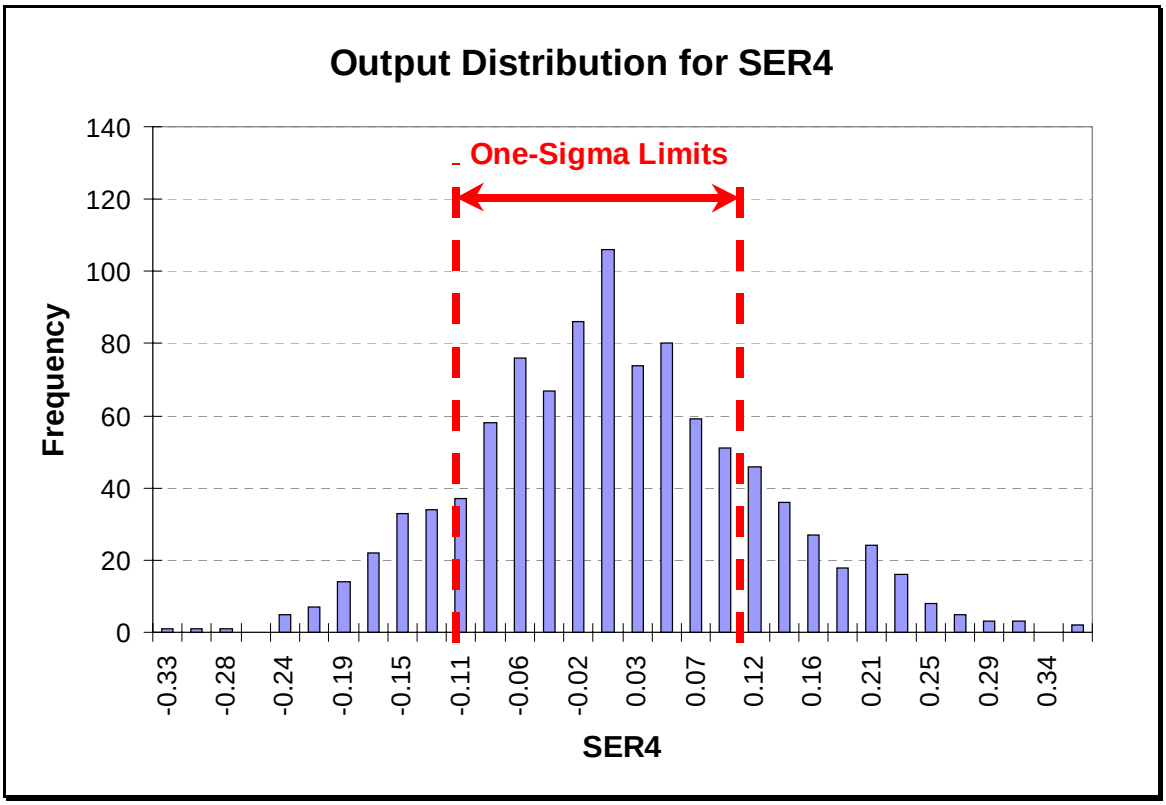


Figure 9-1: Histogram of SER4 for 1,000 Cycles of Monte Carlo, each with 10,000 Samples

The mean SER4 for the 1,000 cycles is very close to zero. The standard deviation of SER4 is about 0.1. The distribution looks a little skewed to the right, but overall is not too far from normal.

Table 9-1: Probability Distributions for Estimating SER1–SER4 at 10,000 Samples

	SER1	SER2	SER3	SER4
Mean (histogram)	0.0003	-0.0002	-0.0005	-0.0001
Stdev (histogram)	0.010	0.015	0.025	0.108
σ_{SER} (equation)	**0.010**	**0.014**	**0.020**	**0.100**

The one-sigma error bounds for the standard errors from the histogram are very close to those predicted by Eq. 9-1 through Eq. 9-4. SER4 is significantly larger that the other standard errors (ten times greater than SER1). To convert the one-sigma bounds on error back to one-sigma bounds on the original moments, Eq. 9-5 below is used.

$$\sigma_{\mu_i} = \sigma_{SERi}\sigma^i \qquad \textbf{Eq. 9-5}$$

Where: σ^i = The i^{th} power of the actual standard deviation of the distribution (if the actual standard deviation is not known, the estimate can be used as long as the sample size is not too small)

The theoretical one-sigma limits versus sample size for the standard errors SER1 through SER4 are shown below in Figure 9-2. The actual errors shown are from two different Monte Carlo simulations of a billion samples. Additionally, the one-sigma bounds on error are shown for comparison.

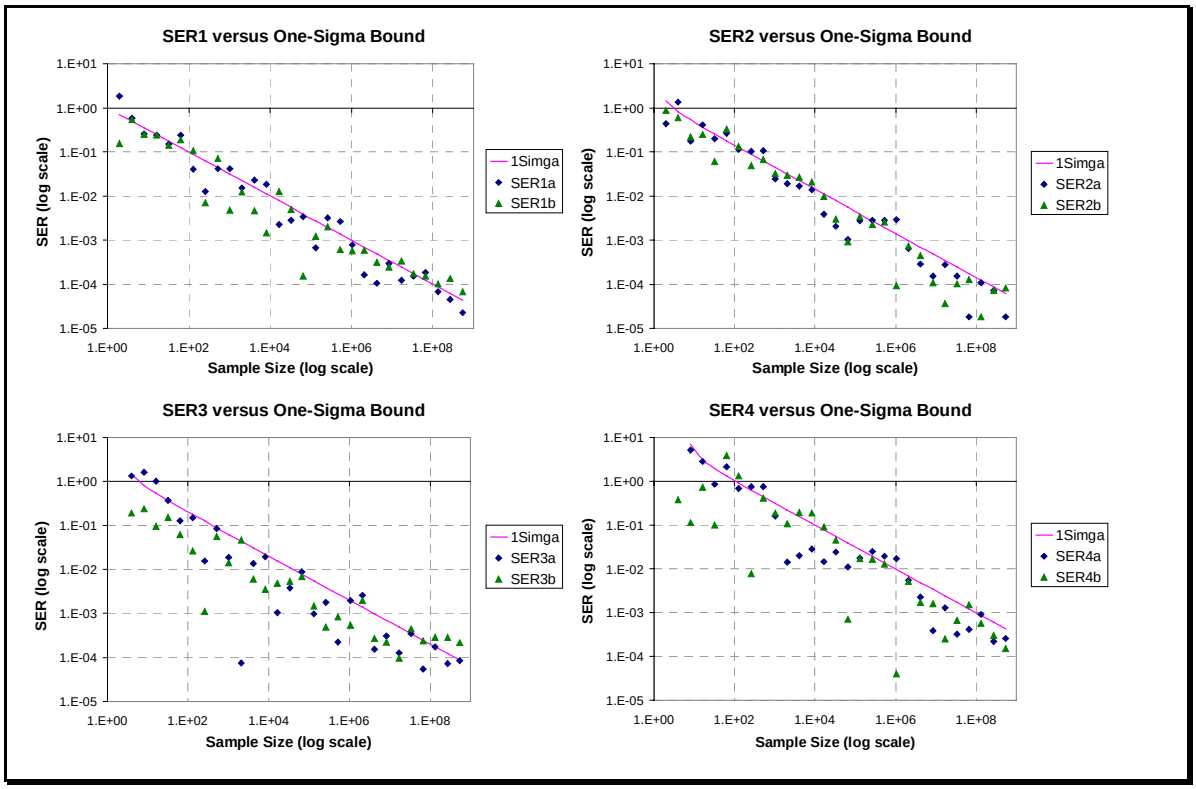


Figure 9-2: Error in Moments versus Sample Size and One-Sigma Bound

The one-sigma bound curves on the graphs are calculated by Eq. 9-1 through Eq. 9-4. Each of the one-sigma bound curves has the same general slope but is shifted up. Figure 9-2 does show that the equations for the one-sigma bounds for SER1 through SER4 do estimate the Monte Carlo error well for the whole range of sample sizes. Having an estimate for the error in the output distribution moments as a function of only the sample size is a valuable tool to determine the number of samples to run. Figure 9-3 below shows these four one-sigma bounds on the same graph to allow for easy comparison.

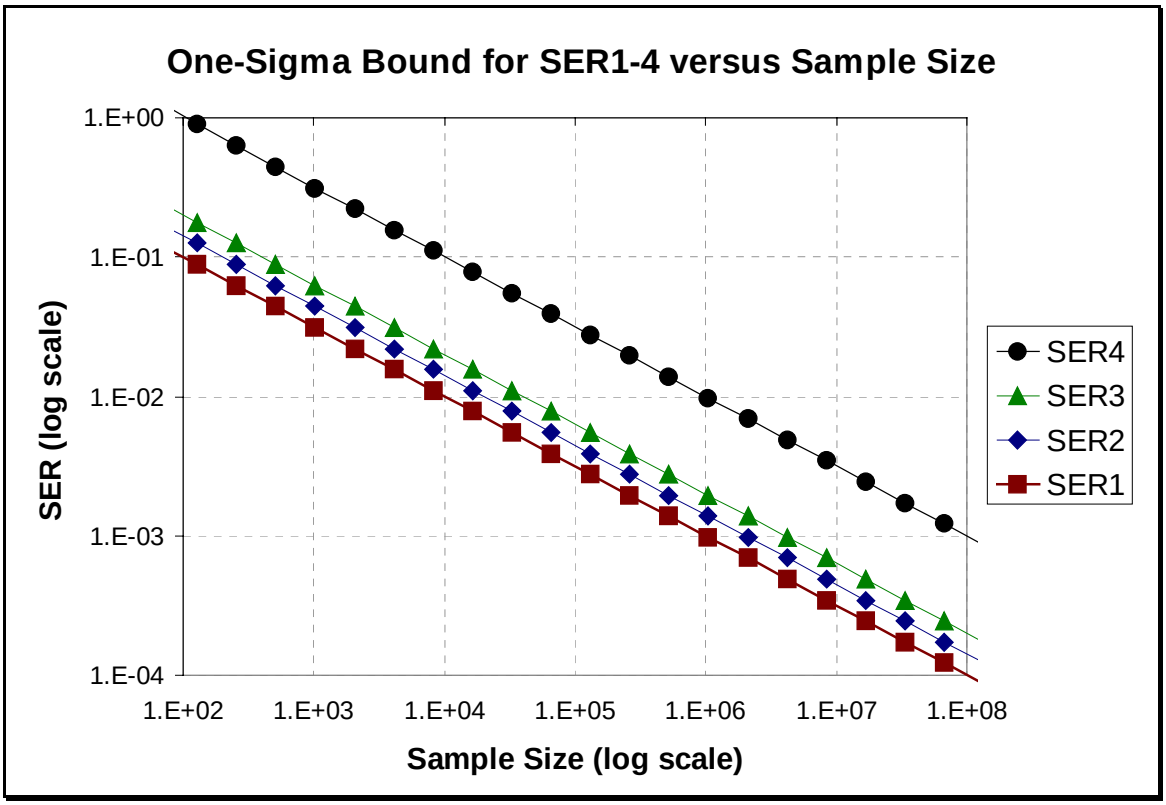


Figure 9-3: The One-Sigma Bound on SER1-4 versus Sample Size

The one-sigma bounds for all four standard moment errors have a very similar slope (decreasing at about the square root of n). But, to achieve a quality level of .001 for SER4, 100 million samples is required, versus one million for SER1. An estimate for the error of the moments can now be determined based on sample size. The estimate (σ_{SER}) can then be combined with the sensitivities for PPM rejects determined in Chapter 8 to estimate the variation in PPM rejects (calculated from the moments) based on sample size. But first, another source of error must be introduced—the error inherent in using the Lambda distribution.

Section 9.3. The Error Inherent in Using the Lambda Distribution

Another source of error in using the output distribution moments to estimate the PPM rejects is the error in fitting a real-world distribution with a Lambda distribution. Figure

7-2 showed that even though the first four moments of a normal distribution can be matched perfectly with the Lambda, there is still error in estimating PPM rejects. The higher moments of the Lambda are not the same as the normal distribution (see Table 7-1). The error in fitting a Lambda distribution to a normal and an exponential distribution is shown below in Figure 9-1. The absolute value of the error is taken in order to put it on a log scale.

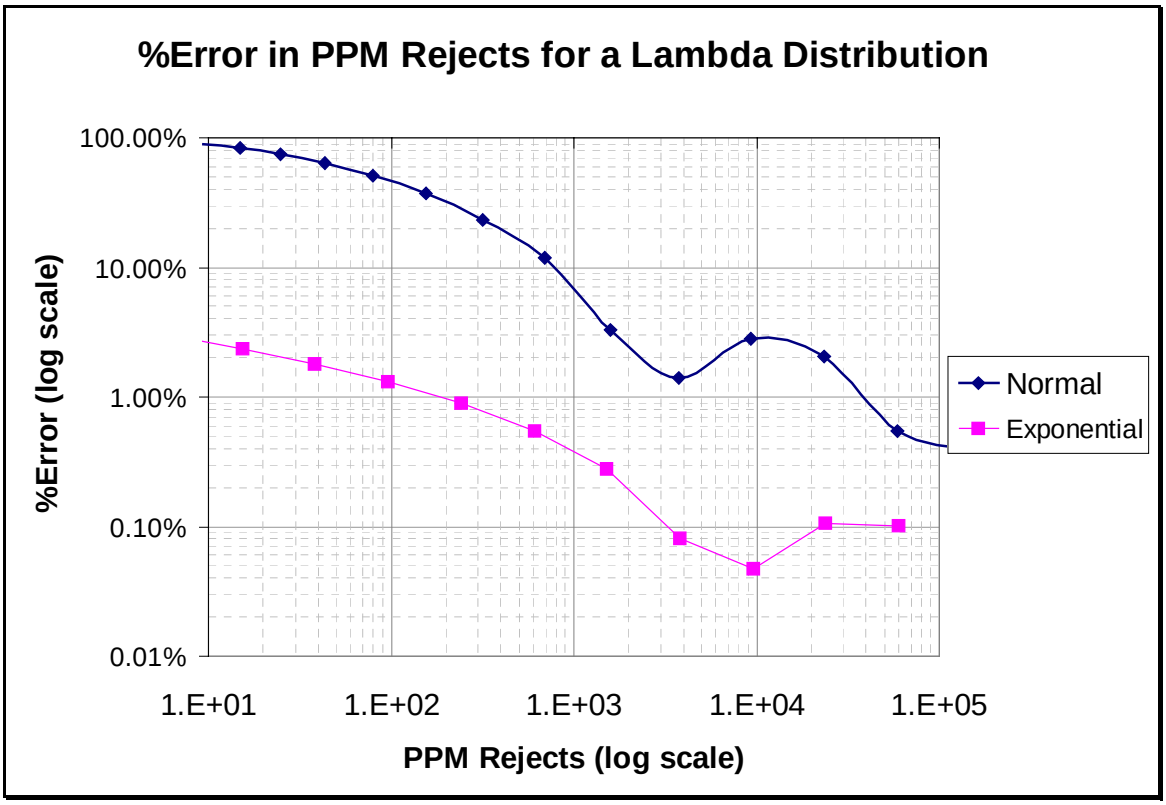


Figure 9-1: Percent Error in PPM Rejects for Fitting a Lambda Distribution to a Normal and Exponential Distribution

For both the normal and the exponential distribution, the Lambda distribution can match the first four moments. But there is error in finding the Lambda parameters, and in the higher moments. The Lambda distribution fits the exponential distribution much better than the normal. The errors above are the absolute value of the errors. The Lambda distribution underestimates the rejects for the normal near the tail, and overestimates the

rejects for the exponential. It is important to note that the error due to fitting a Lambda distribution does not depend on the sample size, but on the quality level or how far out in the tail the specification limits are applied.

Estimating the error introduced through using the Lambda distribution to calculate the percent rejects is not easy. More research should be done to better determine this error, or to determine how to better fit a Lambda distribution to the tails of a distribution when only the distribution moments are known. As an estimate of the percent error inherent in using the Lambda distribution, the error shown above for the normal distribution will be used in the next section.

Section 9.4. Total Error in Estimating PPM Rejects with the Moments and Lambda Distribution

This section will estimate the total error involved in using Monte Carlo simulation with the output distribution moments and the Lambda distribution to calculate the PPM rejects. The total error will be compared to the error in counting the PPM rejects directly. The estimate of error in using the Lambda distribution to model only the first four moments of a distribution can be added to the estimate of the error caused from errors in the moments themselves to get the total error. As in classical error analysis, the variances due to each source of error may be added by root-sum-squares (RSS).

Eq. 9-1 below shows how the two forms of error combine to estimate the total error for the method. $\sigma_{\%PPM,Lambda}$ represents the percent error in PPM rejects introduced by modeling a real distribution with a Lambda distribution. $\sigma_{\%PPM,SER}$ represents the error caused by using estimates of the actual distribution moments.

$$\sigma^2_{\%PPM,Total} = \left(\sigma_{\%ERR,Lambda}\right)^2 + \left(\sigma_{\%ERR,SER}\right)^2$$

$$\sigma_{\%ERR,Total} = \sqrt{\left(\sigma_{\%ERR,Lambda}\right)^2 + \sum_{i=1}^{4}\left(SP_i SER_i\right)^2} \qquad \textbf{Eq. 9-1}$$

The variance due each moment is calculated as the square of the product of its sensitivity percent (SP) and its one-sigma bound on standard moment error. The sensitivity percent for PPM rejects estimates the percent change in PPM rejects due to the standard moment errors (see Figure 8-1). The variance due to the Lambda distribution fit has no sensitivity, as not enough is understood about that source of error. The square of the error displayed for the normal distribution in Figure 9-1 (for the appropriate quality level) will be used for the entire variance due to the Lambda fit. A graphical representation of Eq. 9-1 is shown below in Figure 9-1. Each of the two sources of error plays a major role in the total error, depending on sample size.

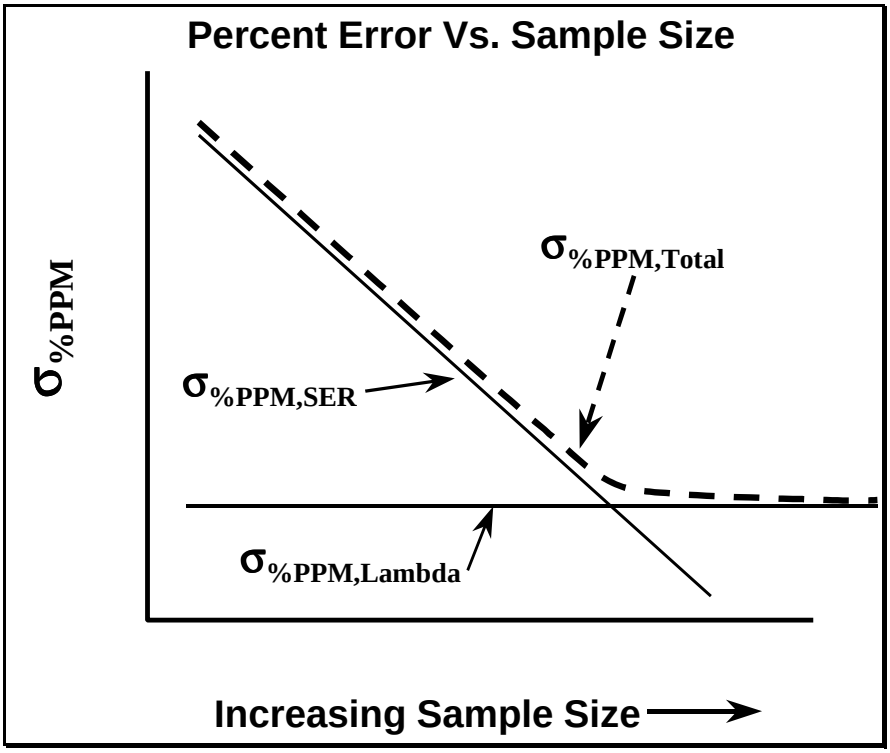


Figure 9-1: Makeup of Error for Estimating PPM Rejects with the Lambda Distribution Versus Sample Size

Because the two sources of error are added as the root-sum-square, the total error is virtually identical to $\sigma_{\%PPM,SER}$ at low sample sizes and $\sigma_{\%PPM,Lambda}$ at larger sample

sizes. The transition between the two errors occurs where the two errors are equal. This figure is drawn with the same format as Figure 9-2 with a log scale.

To calculate the percent error for estimating the PPM rejects for the contact angle of the clutch, the SER for the four moments, the sensitivity of percent error (SP) for each of the moments, and the estimate for $\sigma_{\%PPM,Lambda}$ must be determined. The SER for the four moments can be estimated from the sample size (see Eq. 9-1 through Eq. 9-4). The SP for each of the moments can be estimated by using Figure 8-1 (the sigma level of the specification limits is about three) and Eq. 8-2. Eq. 9-2 below is a summary of calculating $\sigma_{\%PPM,Total}$ for the lower PPM rejects for the clutch problem when using the Lambda distribution.

$$\sigma_{\%PPM,Total}^{\ 2} = \left(\sigma_{ERR,Lambda}\right)^2 + \left[SP_1\left(\sigma_{ERR,SER1}\right)\right]^2 + \left[SP_2\left(\sigma_{ERR,SER2}\right)\right]^2 + \left[SP_3\left(\sigma_{ERR,SER3}\right)\right]^2 + \left[SP_4\left(\sigma_{ERR,SER4}\right)\right]^2 \qquad \textbf{Eq. 9-2}$$

$$= (0.03)^2 + \left[3\left(\sqrt{1/n}\right)\right]^2 + \left[7\left(\sqrt{\frac{2}{n-1}}\right)\right]^2 + \left[2P_3\left(\sqrt{\frac{4}{n-2}}\right)\right]^2 + \left[1.5\left(\sqrt{\frac{100}{n-6}}\right)\right]^2$$

The equation above is used to estimate the one-sigma limit on the error for the lower PPM rejects versus the sample size *n*. From the equation it appears that SER4 contributes the most to the total error, followed by SER2. But at a large number of samples all of the terms get small except for $\sigma_{\%PPM,Lambda}$. This error limit and the actual error for a Monte Carlo simulation are shown below in Figure 9-2.

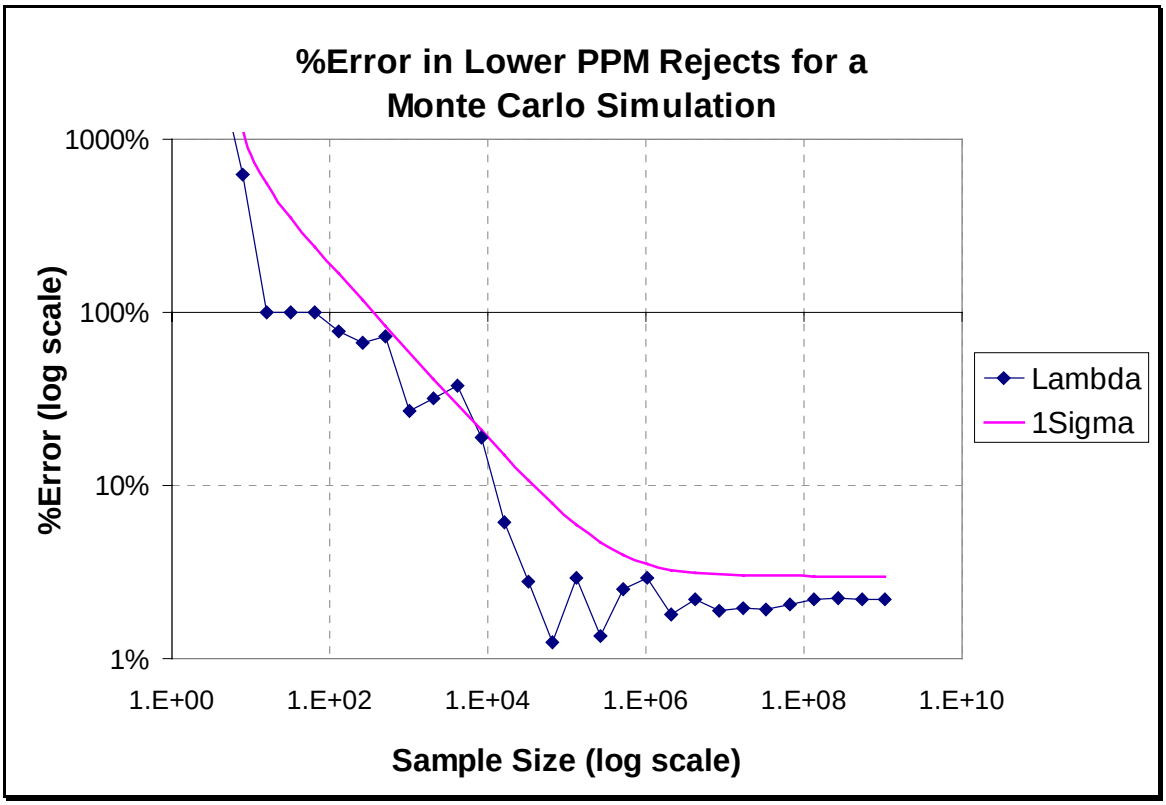


Figure 9-2: Actual Versus Predicted Error for Using Moments and the Lambda Distribution to find Lower PPM Rejects for the Clutch

The one-sigma error limit prediction is quite close to the actual error in PPM rejects. If anything, the one-sigma limit is conservative. With a large sample size the predicted error levels off at three percent due to inadequacy of the Lambda distribution fit. In the above figure, the predicted error limit is a good estimate for the actual error. The estimate of the total error could be improved by a better estimate of $\sigma_{\%PPM,Lambda}$. The next section will show how the error from counting PPM rejects compares to the error using the moments and the Lambda distribution.

Section 9.5. Comparison of Counting Rejects with Using the Lambda Distribution and the Moments

This section will compare two different methods of estimating PPM rejects. One method (count) counts the rejects during the Monte Carlo simulation and estimates the PPM rejects from the counted rejects and the sample size. The second method (Lambda) uses the estimates for the first four output distribution moments to fit a Lambda distribution. Then the Lambda distribution can be used to estimate the PPM rejects for any set of specification limits.

The predicted error for counting rejects was presented in Section 3.5 and Section 9.1. The predicted error for using the Lambda distribution was presented in Section 9.4. Figure 9-1 below compares these two methods together (the actual errors and the predicted one-sigma bounds on the errors).

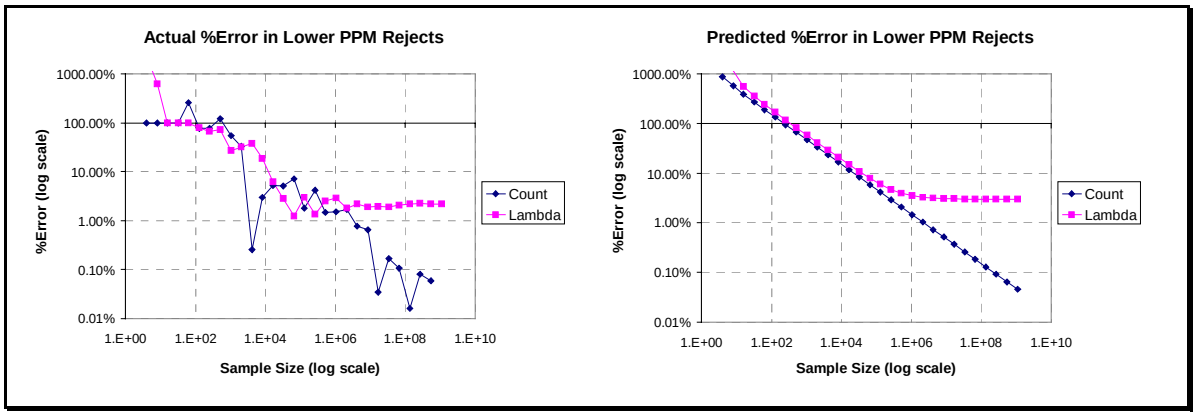


Figure 9-1: Actual Versus Predicted Error for Lower PPM Rejects Using Monte Carlo Simulation

The one-sigma bound for the error by using moments and the Lambda distribution is very close to that from counting the rejects at lower sample sizes. The graph of actual errors looks very much like the predictions except for the addition of random noise.

The actual error using moments and the Lambda distribution to estimate the PPM rejects is not significantly different from the error from counting the rejects at lower sample

sizes. In fact, as long as the Lambda distribution models the real distribution well, there does not appear to be any difference in certainty. The results for upper and total rejects are very similar as well. Figure 9-2 below shows what the predicted error level for lower rejects is at a four-sigma quality level.

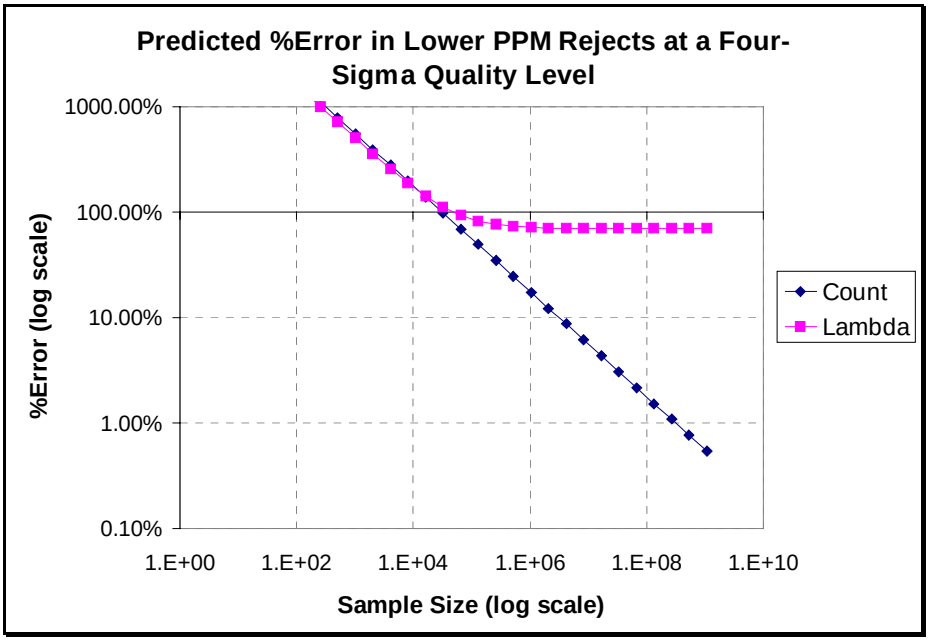


Figure 9-2: Predicted Error in Lower PPM Rejects at a 4-Sigma Quality Level Using Monte Carlo Simulation

At a four-sigma quality level (about 20 PPM rejects for this output distribution) counting rejects does not achieve a better confidence level than using the moments and the Lambda distribution, except at sample sizes greater than 100,000.

According to the model created in this thesis for estimating the confidence interval from using the moments and the Lambda distribution to determine the PPM rejects, the confidence level from the Lambda distribution is not significantly different from counting rejects. The major difference is that the error in modeling a real distribution with a Lambda distribution puts a lower limit on the confidence level according to the sigma-

level of the problem (see Figure 9-1). A better distribution than the Lambda, or a better method of fitting a real distribution in the tails, would help reduce this error.

Section 9.6. Sensitivities and Percent Contribution from Monte Carlo

The estimates for the sensitivities of the objective function to the input variables using Monte Carlo were presented in Section 3.4. The Pearson correlation coefficient (r_p) and the Spearman rank coefficient (r_s) measure the correlation between the output variation and the input variation. The Spearman rank coefficient does not use the actual values of the variables, but only their rank (relative order by size). Using the rank prevents a correlation from being estimated if one does not exist, but loses some information in the process. The Pearson correlation coefficient might predict a correlation when one does not exist.

To calculate the percent contribution from the correlation coefficients, the correlation coefficients are scaled so that the sum of the squares is equal to 100 percent. The percent contribution for a term is then estimated by the square of its scaled correlation coefficient as shown below in Eq. 9-1.

$$\%\text{Contribution}_i = \frac{r_i^2}{\sum_{j=1}^{n} r_j^2} \qquad \textbf{Eq. 9-1}$$

The sensitivity of the objective function can also be estimated from the percent contribution of a variable, the variation of that input variable, and the total variation of the output function. This estimate of sensitivity is assuming a linear assembly function by using Eq. 4-2 and Eq. 4-1. Eq. 9-2 below shows the calculation of the sensitivity from the percent contribution.

$$\text{Sensitivity}_i = \frac{\sqrt{(\%\text{Contribution}_i)\sigma_{Total}^2}}{\sigma_i} \qquad \textbf{Eq. 9-2}$$

If more than the linear sensitivities are desired, correlation coefficients would have to be calculated for the products of the input variables, greatly increasing the complexity. Thus, the estimation for sensitivities and percent contributions for Monte Carlo are not the extremely good, but are rough, linear estimates.

Section 9.7. Quality Loss estimated by Monte Carlo

The Taguchi quality loss function discussed in Section 6.1 and Section 8.3 can be estimated by using the mean and variance estimated by the output distribution obtained from Monte Carlo, or by summing up the average quality loss of each part. Eq. 8-2 and Eq. 8-3 describe the derivatives of the loss function with respect to SER1 and SER2. Using the standard moment errors to estimate the quality loss can yield a confidence level by the following equation.

$$\sigma_{L,Total}^{\ 2} = \left[\left(\frac{\partial L}{\partial SER1}\right)\sigma_{SER1}\right]^2 + \left[\left(\frac{\partial L}{\partial SER2}\right)\sigma_{SER2}\right]^2 \qquad \textbf{Eq. 9-1}$$

$$= \left[\left(2K(\hat{\mu}_1 - m)\sqrt{\hat{\mu}_2}\right)\left(\sqrt{\frac{1}{n}}\right)\right]^2 + \left[\left(K\hat{\mu}_2\right)\left(\sqrt{\frac{2}{n-1}}\right)\right]^2$$

As shown in the equation above, the error in estimating the quality loss is inversely proportional to the square root of *n* (the sample size). Other sources of error for estimating the quality loss are the error in the cost constant and the error in estimating the cost by a quadratic function. These errors should be evaluated first, because reducing the error from the Monte Carlo simulation (given in Eq. 9-1) beyond the error in the cost model itself is fruitless. The actual versus one-sigma bound on the percent error for quality loss is shown below in Figure 9-1.

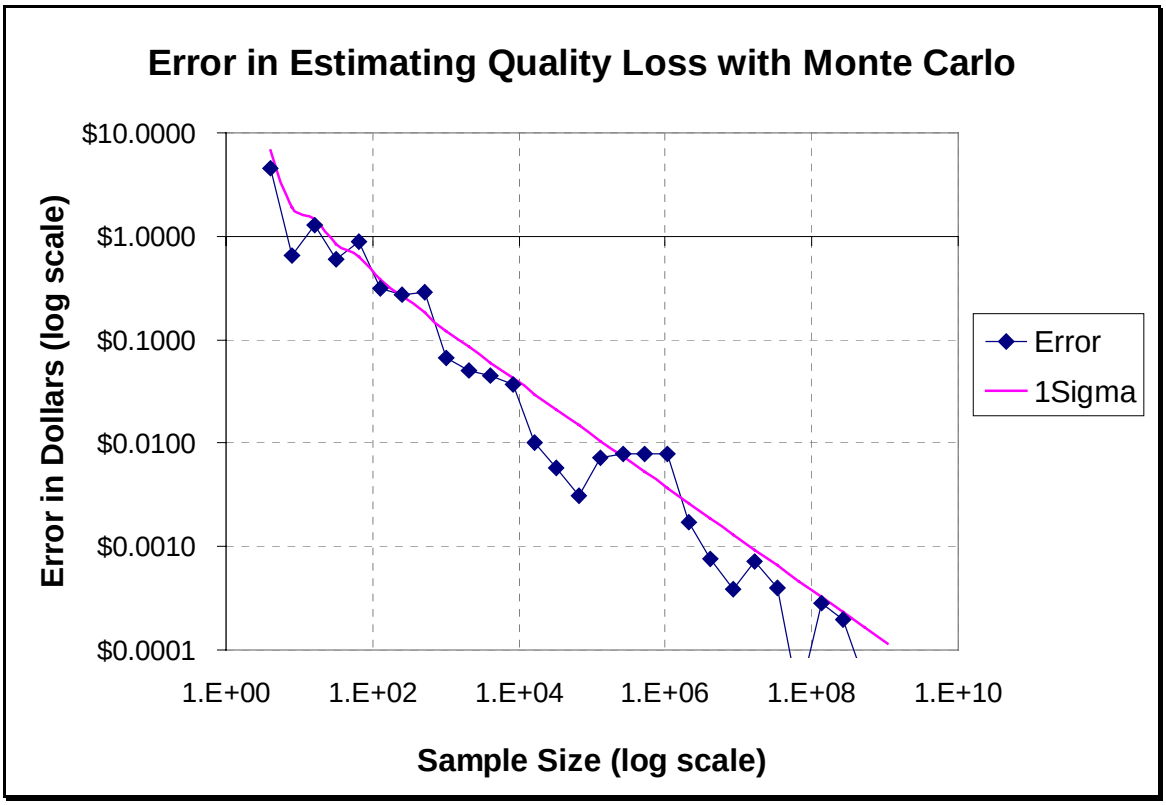


Figure 9-1: Percent Error Estimating Quality Loss with Monte Carlo Simulation

The error estimating the quality loss with Monte Carlo decreases with the square root of the sample size. The estimate of the distribution mean and variance are used to calculate the one-sigma bound, so the one-sigma curve is not completely smooth at small sample sizes. To determine the contribution to quality loss from each of the input variables would require calculating a correlation coefficient and percent contribution for the inputs. Thus, Monte Carlo can easily calculate the quality loss, but determining the percent contribution to the quality loss is not as straightforward.

Section 9.8. Strengths/Weaknesses Summary for Monte Carlo

Monte Carlo simulation is a robust, realistic simulation method, useful for many variation analysis situations. Monte Carlo is useful for calculating PPM rejects and quality loss. Using Monte Carlo, an engineer can either count rejects during the simulation, and use

them to estimate the PPM rejects; or the output distribution moments can be used to fit a distribution to estimate the rejects. Using the Lambda distribution with the moments was determined to be as good as counting rejects for sample sizes up to a million, limited by the accuracy of the Lambda distribution to model the real output distribution (mismatch in the fifth moment and greater).

Monte Carlo does not as easily calculate the percent contribution and sensitivities of the input variables. The calculation is either not robust, or not as accurate as other methods that will be discussed later. But the estimates for the sensitivities can be gotten while using Monte Carlo, if they are desired.

Monte Carlo is extremely robust. The assembly function for Monte Carlo can be explicit, implicit, or even discontinuous. The input variables can be correlated (not independently random) or even discrete. The variables can be a function of time, letting time change during the simulation. Monte Carlo is very flexible to allow for unusual situations.

The principle complaint about Monte Carlo is the inefficiency due to large sample sizes, making it tedious for design iteration. Also, for implicit assembly functions, which frequently occur in mechanical assemblies and mechanisms, the necessity of performing an iterative solution for each simulated assembly decreases the efficiency by an order of magnitude.

Chapter 10. Estimating the Effectiveness of the MSM and RSS

The method of system moments and RSS first order approximation will be evaluated in the same chapter, because they are similar (a quadratic and a linear model). The main contributions will be in estimating the accuracy of the methods in order to determine if they can be used. The method of system moments (MSM) will be evaluated first.

Section 10.1. The Method of System Moments

The method of system moments uses the distributions for the input variables, along with their sensitivities (relative to the desired output function) to estimate the output distribution moments. Thus, the errors involved with MSM come from the errors in the input variable distribution moments (discussed in Section 8.3), the sensitivities, and the non-quadratic nature of the assembly function.

10.1.1. Finding the Sensitivities of the Assembly Function to the Input Variables

The sensitivities of the assembly function with respect to the input variables can be calculated from the partial derivatives of the assembly function. Three other methods to obtain these sensitivities are matrix algebra, finite difference derivatives, and response surfaces.

10.1.1.1. Taking the Derivatives of the Assembly Function

In Section 4.4, the first and second partial derivatives for the contact angle of the clutch assembly were found from the explicit assembly equation. Most assembly functions are difficult to express in explicit form, and most of the others are difficult to differentiate. In general, taking the derivatives directly from the assembly function is impractical.

10.1.1.2. Sensitivities by Matrix Algebra

In Section 5.5, the first partial derivatives were found by linearizing the implicit assembly equations and solving for the sensitivities by matrix inversion. The clutch assembly problem was also demonstrated in that section. This method is easily automated, and is currently used in the CATS software. The sensitivities obtained through linearization and matrix algebra have been shown to be the same as the explicit derivatives (compare Table 5-1 and Eq. 4-1). But, in linearizing the assembly function, higher order terms are neglected. Valuable information could be lost if the assembly function is highly non-linear.

10.1.1.3. Finite Difference Derivatives

One method to obtain close approximations of the partial derivatives is the finite difference method. Finite difference methods can easily find the first and second derivatives of any function by evaluating the function while changing one or two input variables at a time. There are many established methods to perform the finite difference derivatives, but these will not be evaluated by this thesis. The objective of a finite difference derivative is to most accurately estimate the actual partial derivatives.

10.1.1.4. Response Surfaces

Section 6.8 discusses how a response surface can be used to estimate the assembly function as a quadratic function. The partial derivatives of the quadratic function can then be easily found, and evaluated at any value of the input variables. But how do these sensitivities compare to the exact or finite difference derivatives? The process to investigate the effect of different sensitivities on the output moments from MSM is shown below:

- The sampling interval for the central composite design (CCD) for each variable was calculated to be a certain sampling ratio times the standard deviation for that input variable.
- A quadratic response surface was fit to the data from the CCD.
- The derivatives of the quadratic equation were taken easily, as demonstrated by Eq. 6-2.
- The sensitivities and input variable information were used with the method of system moments to estimate the first four output distribution moments.
- The standard moment errors for the output moments were calculated using the benchmark Monte Carlo estimates at one billion samples.

Figure 10-1 below shows the standard moment errors for the method explained above versus the sampling ratio, and how calculating the sensitivities using a response surface at different points can increase the accuracy of the method of system moments.

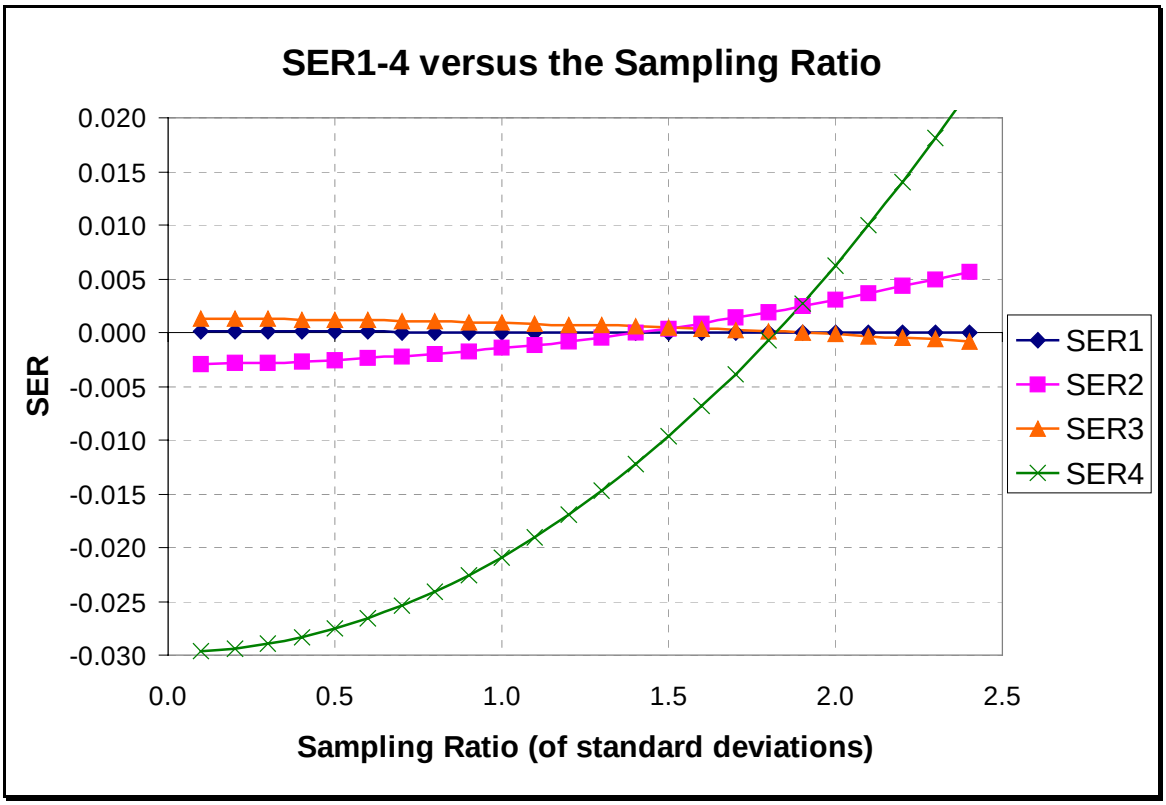


Figure 10-1: Comparing the Accuracy of MSM using a Response Surface to find the Sensitivities

The results shown in Figure 10-1 above were obtained by using a central composite design (see Section 6.8) response surface to find the sensitivities for the method of system moments. The levels for the input variables were set to plus and minus the standard deviations of the variables times the "sampling ratio." The standard moment errors at small sampling ratios approach the errors using the exact partial derivatives (zero sampling ratio). The results from using the exact derivatives (see Table 4-1), and the results of using a response surface at a few key points, are shown below as standard moment errors.

Table 10-1: Comparison of Moment Errors due to Sensitivities Obtaines by Partial Derivatives and Response Surfaces

	Partial Derivatives	Response Surface 0.1σ	1.4σ	1.8σ
SER1	7.33E-05	7.30E-05	1.88E-05	-8.47E-06
SER2	-2.94E-03	-2.92E-03	-3.47E-05	1.88E-03
SER3	1.28E-03	1.27E-03	5.78E-04	1.16E-04
SER4	-2.98E-02	-2.97E-02	-1.23E-02	-6.96E-04

The result of performing the MSM on the response surface using the sampling ratio of only 0.1 times the standard deviations for the input variables approaches the standard moment error for using the exact derivatives. The ratio of 1.4 is close to the point where the error in variance (SER2) in minimized. The ratio of 1.8 is close to the point where the kurtosis (SER4) is minimized. Any sampling ratio less than about 2.0 results in better estimates for each of the output distribution moments than using the exact partial derivatives.

10.1.2. Estimating the Error from Modeling the Assembly Function as Quadratic using the Monte Carlo Difference Method

A quadratic variation model is often very accurate. Trying to estimate its accuracy would normally require a very large Monte Carlo sample size. For example, using the response surface sensitivities at a sampling ratio of 0.1 (or the actual partial derivatives), the largest of the standard moment errors was for the kurtosis, -0.0297 (see Table 10-1). Using the equations developed for the standard moment errors versus sample size (see Eq. 9-1 through Eq. 9-4), the one-sigma bound on SER1-4 can be estimated. Using the same equations, the sample size for a certain one-sigma confidence interval can also be determined. Therefore, Table 10-1 below shows the sample size needed to conclude (with only a 68 percent confidence level) that the moments calculated by MSM (using sensitivities from a response surface at a sampling ratio of 0.1) are not accurate.

Table 10-1: Required Sample Size to for a One-Sigma Confidence Level on the Error of MSM (Sampling Ratio = 0.1)

	Mean	Variance	Skewness	Kurtosis
Actual SER	7.30E-05	-2.92E-03	1.27E-03	-2.97E-02
Sample Size for one-sigma confidence	190,000,000	240,000	2,500,000	110,000

The accuracy of MSM is so great that at least 110,000 samples of Monte Carlo would be required to even suspect an error in one of the moments. These calculations assume that Monte Carlo predicts the exact value of the output distribution moments, and the standard moment errors are the one-sigma bound for error for each of the moments. Even if the actual assembly function were quadratic, so that MSM would result in the exact answer, the same sample sizes above would be necessary to confirm it with Monte Carlo for the same level of certainty. Clearly a better method is needed to estimate the accuracy of MSM because using a larger sampling ratio can even further enhance the accuracy of MSM.

The method presented by this thesis to estimate the error of MSM is the Monte Carlo difference method (MCD). This method uses Monte Carlo with both the actual assembly function and the quadratic approximation of the assembly function. The following are the steps of the MCD method:

- Determine the quadratic approximating equation by putting the sensitivities into the Taylor series expansion, or by using the response surface equation.

- For each sample of Monte Carlo, calculate the value of both the actual assembly function and the quadratic approximating equation.

- Once the desired sample size is reached, calculate the standard moment errors for the two sets of responses, assuming the moments from the actual assembly function are the true values.

- The estimates of the standard moment errors can then be used to approximate the errors using MSM with the quadratic approximating equation.

This accuracy of the Monte Carlo difference method definitely depends on the sample size used. The reason that the MCD method can better estimate the errors is that it uses the same data points (set of input variable values) to estimate the moments of both equations (the actual and the quadratic approximate). Figure 10-1 below shows the estimates of SER1-4 that are obtained as the sample size increases.

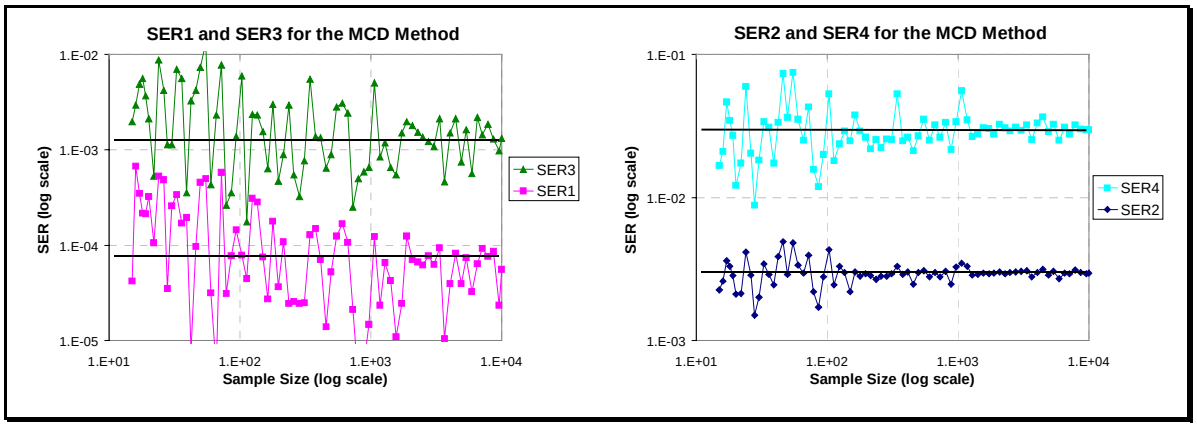


Figure 10-1: Estimating SER1-4 Using the Monte Carlo Difference Method

The dark horizontal lines show the exact standard moment errors. Because the actual errors for variance and kurtosis are greater than the other two errors, the MCD method better estimates their standard moment errors. After only about 500 samples, the estimate of SER2 is very good, and the estimate of SER4 is the same order of magnitude. For the other two standard errors (SER1 and SER3), even at about 10,000 samples the MCD method can estimate the order of magnitude of the error. Of course, if the actual assembly function is a quadratic, the quadratic approximating function should be exact. In that case,

at any sample size, the predicted standard moment error should be zero (assuming no round-of error).

What is not apparent from the figure above is that the MCD method estimates the magnitude and the sign of SER1-4 (the absolute values of the errors were plotted in order to use a log scale). Therefore, the estimates of error can be used to correct the moments predicted by MSM to be more accurate. This results in a hybrid method that is more accurate that Monte Carlo alone. This will be discussed in detail in Section 12.3 as part of the Random Response Surface method.

10.1.3. Strengths/Weaknesses Summary for MSM

The method of system moments is a valuable variation analysis method that can estimate the first four output distribution moments. The method of system moments requires the first eight distribution moments for each of the input variables and the first and second-order sensitivities of the response function. The accuracy of the method of system moments depends on how well a quadratic function can approximate the true assembly function, and how that approximating function was obtained (response surface or finite-difference). The Monte Carlo Difference method presented in Section 10.1.2 is one way to determine the accuracy of the MSM method with much fewer samples than using Monte Carlo alone. More verification of the Monte Carlo Difference method will be presented in Section 12.3.

Most variation analysis problems can be fit well by a quadratic function because the tolerances are relatively small. Some problems can even be approximated well by a linear analysis method. The following section will discuss RSS first-order approximation.

Section 10.2. RSS First-order Approximation

In some situations, the magnitude of the parts per million rejects is more important that the exact number. For example, a three-sigma process is one for which the distance from

the mean of the output distribution to either of the specification limits is three times the standard deviation. Table 10-1 below shows the PPM rejects that would be expected from a normal distribution at different sigma quality levels.

Table 10-1: Total PPM Rejects for a Normal Distribution versus the Sigma Quality Level

σ Level	Total PPM Rejects
1	317,311
2	45,500
3	2,700
4	63
5	0.6

Because the output distributions for most assembly functions tend to be normal, the table above is a quick reference for the magnitude of the rejects that would be expected. If the specification limits are centered around the nominal, an estimate of the standard deviation of the output distribution is all that is needed to get the magnitude of the rejects. In some cases, it does not matter if the number of rejects is 2,700 or 3,000 (about a ten percent difference), but rather that the number of rejects is not more than 5,000. All variation analysis problems have some level of uncertainty in them anyway.

Chapter 8 and Section 9.3 discussed how errors in estimating the output distribution moments and fitting a Lambda distribution can each add error to an analysis. Large sample sizes are required for Monte Carlo to reduce the error, even when counting rejects. Section 8.3 found that poor estimates of the kurtosis for the input variable distributions could be a major source of error. If the input information (input distribution moments, sensitivities, all of the sources of variation in an assembly, exact location of the specification limits for good performance, etc.) is not very reliable, then RSS first-order approximation could be the best choice of analysis method. In such cases, no number of samples from Monte Carlo can increase the certainty of the analysis over a simple RSS first-order approximation.

Additionally, if the objective of the variation analysis is to estimate the cost in terms of the Taguchi quality loss function (see Section 6.1) only the mean and the variance of output distribution are required. So how can the accuracy of the RSS method be estimated? The state-of-the-art is to perform a Monte Carlo simulation (with a large sample size) or MSM and see if the nonlinear effects are sufficient to change the results. But performing one of these other methods cancels the benefits of performing a linear analysis. Another way would be to compare the tolerances to the first and second derivatives of the assembly function.

10.2.1. Estimating the Error in Assuming a Linear Assembly Function (Linearization Error)

Eq. 4-1 showed that the quadratic estimate for the variance of the output is a function of the first, second, and cross derivatives, as well as the first four input distribution moments. In order to see if a linear model is valid, the quadratic equation will be simplified to estimate the error of assuming a linear assembly function. Each of the input variables will be analyzed independently (holding the other variables constant) to estimate its quadratic versus linear effect on the first four moments of the output distribution (like a partial derivative). Because only one variable is investigated at a time, the expressions simplify down and are quite usable.

The error in assuming the assembly function is linear instead of quadratic will be called "linearization error." Eq. 10-1 below shows how Eq. 4-1 (quadratic estimate of the mean) is transformed (to a form like the standard moment error) if only one variable contributes variation to the assembly function. Refer to Eq. 4-2 for a definition of the variables.

$$\mu_{1,Q} - \mu_{1,L} = \sum_{i=1}^{n} b_{ii} = b_{aa} \qquad \textbf{Eq. 10-1}$$

$$\left(\frac{\mu_{1,L} - \mu_{1,Q}}{\sigma_L} \right)_a = -\frac{b_{aa}}{\sqrt{(b_a)^2}} = -\frac{b_{aa}}{|b_a|}$$

Where: $\mu_{1,Q}$ = The quadratic estimate of the mean of the assembly function

$\mu_{1,L}$ = The linear estimate of the mean of the assembly function

σ_L = The linear variation of the assembly function, but only for the one variable (to approximate the true variation)

The final left side of Eq. 10-1 ends up being an approximation for the standard moment error of the mean (SER1) for the assembly function. This estimate of standard moment error will be referred to as SER1a to differentiate it from SER1, the error from the full analysis. Eq. 10-2 below shows how the formula is simplified to the ratio of the second derivative and derivatives of the assembly function times one-half the standard deviation of the input variable under question. This ratio will be referred to as the quadratic ratio (QR).

$$SER1a \approx -\frac{b_{aa}}{|b_a|} = -\frac{\frac{1}{2}\frac{\partial^2 f}{\partial a^2}\sigma_a}{\left|\frac{\partial f}{\partial a}\right|} = -QR_a \qquad \textbf{Eq. 10-2}$$

Where: SER1a = The estimate of SER1 that is obtained by assuming a quadratic function is linear (only function of variable *a*), or linearization error due to variable *a*.

The quadratic ratio of the assembly function with respect to an input variable estimates the magnitude of the standard error of the mean that is caused by that input variable when assuming the assembly function is linear instead of quadratic (linearization error).

To use the quadratic ratio to estimate the linearization error, generally the top few contributors to the linear assembly variance should be checked. Additionally, the variable with the largest standard deviation could also be checked, as the quadratic ratio is proportional to the standard deviation of the input variable. For example, Table 5-2 in Section 5.4 shows the percent contribution for the three input variables for the clutch assembly. The hub radius (*a*) not only has the largest standard deviation (by a factor of four), but it also contributes to over 80 percent of the total variation. Thus, it would be the first input variable to be checked for the size of its quadratic ratio.

To estimate the standard moment error for the variance, skewness, and kurtosis that is caused by linearization, the quadratic estimate for the moments mentioned in Section 4.1 and in Eq. 4-1 are presented below. Eq. 10-3 below assumes the assembly is only a function of one variable (*a*).

$$\left(\mu_{2,Q}\right)_a = b_a^2 + 2b_a b_{aa}\alpha_3 + b_{aa}^2\alpha_4 - b_{aa}^2 \qquad \textbf{Eq. 10-3}$$

$$\left(\mu_{3,Q}\right)_a = b_a^3\alpha_3 + b_{aa}^3\alpha_6 + 3b_a^2 b_{aa}\alpha_4 + 3b_a b_{aa}^2\alpha_5 - 3b_{aa}\left(b_a^2 + 2b_a b_{aa}\alpha_3 + b_{aa}^2\alpha_4\right) + 2b_{aa}^3$$

$$\left(\mu_{4,Q}\right)_a = b_a^4\alpha_4 + b_{aa}^4\alpha_8 + 4b_a^3 b_{aa}\alpha_5 + 4b_a b_{aa}^3\alpha_7 + 6b_a^2 b_{aa}^2\alpha_6 - 4b_{aa}\left(b_a^3\alpha_3 + b_{aa}^3\alpha_6 + 3b_a^2 b_{aa}\alpha_4 + 3b_a b_{aa}^2\alpha_5\right) + 6b_{aa}^2\left(b_a^2 + 2b_a b_{aa}\alpha_3 + b_{aa}^2\alpha_4\right) - 3b_{aa}^4$$

Where: $\alpha_i = \dfrac{\mu_i}{\sigma^i}$ = The i[th] standardized moment for the input distribution of *a*

σ^i = The standard deviation input variable (*a* in this case) to the i[th] power

From the set of equations above, it can be noted that because only one input variable is being examined at a time, the first term on the right of each equation is the linear estimate for that same moment. Thus, subtracting that term, dividing by the appropriate power of the linear estimate of the standard deviation of the output function, and collecting terms results in Eq. 10-4 below.

$$\left(\frac{\mu_{2,L} - \mu_{2,Q}}{\sigma_L^2}\right)_a = -\frac{2b_a b_{aa}\alpha_3 + b_{aa}^2\left(\alpha_4 - 1\right)}{b_a^2} \qquad \textbf{Eq. 10-4}$$

$$\left(\frac{\mu_{3,L} - \mu_{3,Q}}{\sigma_L^3}\right)_a = -\frac{b_{aa}^3\left(\alpha_6 - 3\alpha_4 + 2\right) + b_a^2 b_{aa}\left(3\alpha_4 - 3\right) + b_a b_{aa}^2\left(3\alpha_5 - 6\alpha_3\right)}{\left(b_a^2\right)^{\frac{3}{2}}}$$

$$\left(\frac{\mu_{4,L} - \mu_{4,Q}}{\sigma_L^4}\right)_a = -\frac{b_a^3 b_{aa}\left(4\alpha_5 - 4\alpha_3\right) + b_a^2 b_{aa}^2\left(6\alpha_6 - 12\alpha_4 + 6\right)}{b_a^4} - \frac{b_a b_{aa}^3\left(4\alpha_7 - 12\alpha_5 + 12\alpha_3\right) + b_{aa}^4\left(\alpha_8 - 4\alpha_6 + 6\alpha_4 - 3\right)}{b_a^4}$$

The set of equations above can now easily be put into a form similar to Eq. 10-2, in order to approximate the standard moment errors for variance, skewness and kurtosis. The only

difficulty is when a term in the numerator has a b_a to an odd power, the sign of b_a must be retained. In those cases the extra variable (S) is used where the sign of the first derivative is required. The resulting expressions are found below in Eq. 10-5.

$$SER2a \approx -2S\alpha_3 QR - (\alpha_4 - 1)QR^2 \qquad \textbf{Eq. 10-5}$$
$$SER3a \approx -3(\alpha_4 - 1)QR - 3S(\alpha_5 - 2\alpha_3)QR^2 - (\alpha_6 - 3\alpha_4 + 2)QR^3$$
$$SER4a \approx -4S(\alpha_5 - \alpha_3)QR - 6(\alpha_6 - 2\alpha_4 + 1)QR^2$$
$$- 4S(\alpha_7 - 3\alpha_5 + 3\alpha_3)QR^3 - (\alpha_8 - 4\alpha_6 + 6\alpha_4 - 3)QR^4$$

Where: S = The sign of b_a, or also the sign of the first derivative of the assembly function with respect to the input variable being analyzed

These approximations for the standard moment error due to linearization can even further be simplified if the input variable is normally distributed. The third through the eighth standardized moments of a normal distribution are 0, 3, 0, 15, 0, and 105. The estimates for standard moment error from Eq. 10-2 and Eq. 10-5 can thus by simplified for a normal input variable as shown below in Eq. 10-6.

$$SER1a \approx -QR \qquad \textbf{Eq. 10-6}$$
$$SER2a \approx -2QR^2$$
$$SER3a \approx -6QR - 8QR^3$$
$$SER4a \approx -60QR^2 - 60QR^4$$

10.2.2. Estimating the Linearization Error for the Clutch Assembly

For the clutch assembly, it has already been pointed out that the hub radius (a) not only had the greatest standard deviation of the input variables, but it also contributes over 80 percent of the total variation of the contact angle. The first step in the process would be to find the first and second derivatives of the assembly contact angle with respect to the hub radius.

Finding the derivatives could be done by any of the methods already mentioned in this thesis (taking the derivatives of the explicit function, finite difference methods, response surface methods, etc.) Then the quadratic ratio could be calculated. All of the information

necessary to estimate the linearization error (and to determine the actual linearization error) has already been calculated in earlier chapters. The explicit derivatives were calculated in Section 4.4. Table 2-2 contains the input variable standard deviations, Table 4-1 contains the results of linear analysis, and Table 5-3 contains the results from the quadratic analysis. Table 10-1 below shows the input data and calculation of the quadratic ratios for the three variables in the clutch assembly.

Table 10-1: Calculations for the Quadratic Ratios of the Variables for the Clutch Assembly

	a	***c***	***e***
1st Derivative	-11.91	-23.73	11.82
2nd Derivative	-20.11	-81.05	-20.41
Standard Deviation	0.01666	0.00333	0.00416
S **(sign of b_i)**	**-1**	**-1**	**+1**
QR (quadratic ratio)	**-0.0141**	**-0.0057**	**-0.0036**

The quadratic ration for the hub radius, *a*, is almost three times larger that the others. Therefore as it also contributes most of the variation to the contact angle, the hub radius should provide a good estimate of the linearization error. Table 10-2 show the estimate of linearization error (in terms of standard moment error) using the QR for each of the input variables.

Table 10-2: Estimates for the Linearization Error from Each of the Three Input Variables of the Clutch Assembly

	a	***c***	***e***
SER1a,c,e	**0.0141**	0.0057	0.0036
SER2a,c,e	**-0.0004**	-0.0001	0.0000
SER3a,c,e	**0.0844**	0.0341	0.0216
SER4a,c,e	**-0.0119**	-0.0019	-0.0008

The linearization errors from the hub radius, *a*, are much greater than from the other variables. Table 10-3 below shows how the estimates of linearization error from the quadratic ratio of ***a*** compares to the standard moment errors of the RSS method (error with respect to the MSM quadratic approximate and the Monte Carlo benchmark moments).

Table 10-3: Summary for Estimates of Linearization Error for the Clutch Assembly

	Estimates Obtained From:		
	Quadratic Ratio of ***a***	RSS and MSM	RSS and MC Benchmark
SER1	0.0141	0.0156	**0.0157**
SER2	-0.0004	-0.0004	**-0.0034**
SER3	0.0844	0.0936	**0.0944**
SER4	-0.0119	-0.0144	**-0.0441**

The quadratic ratio (QR) from variable ***a*** is a good estimate for the linearization error of the mean and skewness. The actual standard moment error for the variance and kurtosis is significantly greater than predicted by the quadratic ratio, but in that case the quadratic estimate from MSM also has significant error. Overall, the quadratic ratio does estimate the error between the RSS method and the MSM method for the clutch assembly.

Apparently the sign of the standard moment error can even be estimated from the quadratic ratio. The reason for the good results when using the hub radius is that it contributes most to the total variation of the contact angle. In general, probably no single input variable will estimate the error well, but rather give an indication of the magnitude of the error. Figure 10-1 below shows how the linearization error, estimated by the quadratic ratio, can be used to determine how small the variation of an input variable should be in order to assume the assembly function is linear for that variable.

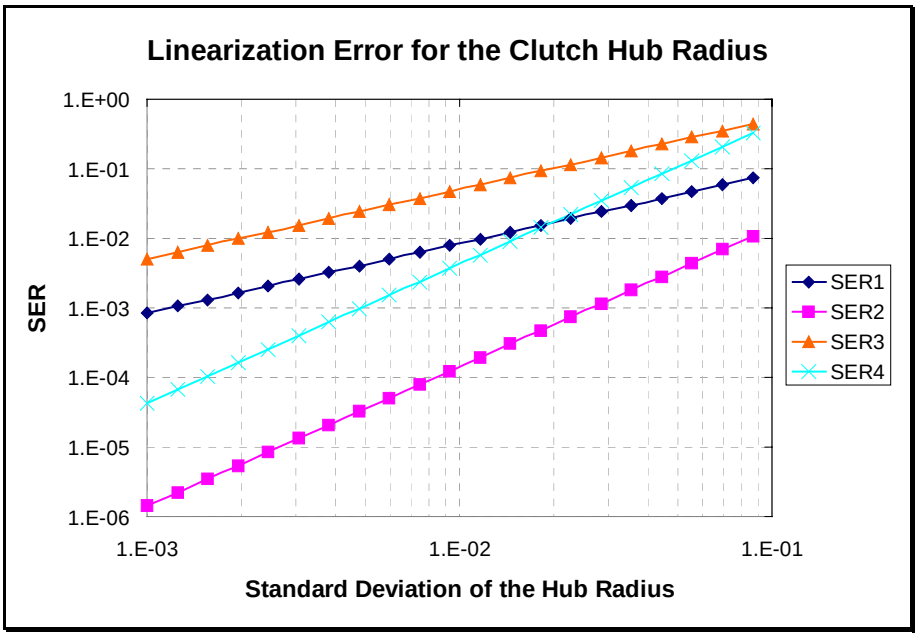


Figure 10-1: Linearization Error from the Hub Radius (determined by the QR) versus the Standard Deviation

Estimating the linearization error using the quadratic ratio can help determine if the variations of the input variables are small enough to assume a linear assembly function. An even better estimate of the linearization error than used above would be to use better estimates for the sensitivities. Section 10.1.1.4 discusses how using the sensitivities from a response surface can significantly improve the estimate of second-order MSM. If the quadratic estimate of the moments is better, than the estimation of the linearization will also be closer.

10.2.3. Advantages of Using a Linear Model

A linear variation analysis model is extremely simple to work with. Estimating a new assembly variance with a change in an input distribution variance is easy arithmetic. Therefore, many different scenarios can be analyzed quickly, making RSS the preferred method for design iteration and optimization.

10.2.4. Summary of Linear Variation Analysis

One obvious weakness of the linear RSS method is that it approximates the true assembly function as linear. This approximation simplifies the analysis making it easier to understand and explore, but it might also reduce the accuracy of the analysis significantly. The key to estimating the error introduced into the analysis by linearizing the assembly function is the quadratic ratio (see Eq. 10-2). Finding the quadratic ratio for the input variables that contribute most of the linear variation will help determine if their standard deviations are small enough relative to their sensitivities to safely assume linearity. More research should be done to determine if the quadratic ratio (or linearization error) from multiple variables can add in some way to better approximate the total linearization error.

Additionally, if there is already significant error in the analysis problem (uncertainty of the input distributions, specification limits, causes of variation, quality loss model, etc.) assuming the assembly function is linear might be the best choice. If a more accurate analysis method were used, the results would probably still not be any more confident than if a linear analysis were used (see Chapter 13).

Chapter 11. Estimating the Effectiveness of the Design of Experiments and Response Surface Methods

This chapter will investigate the accuracy and efficiency of Design of Experiments. A method for using the quadratic ratio in choosing an appropriate experimental design will also be presented.

Design of experiments can be performed by either physically building parts to assemble and test, or by simulating the assembly with a mathematical model. Physically building parts has benefits in that additional sources of variation can be examined. The physical assemblies can be tested for wear, ease of assembly, environmental effects, and other noise for which a mathematical model is not well defined.

Section 11.1. Performing Physical Experiments versus Computer Simulation

In order to have accurate estimates for the analysis, the dimensions of the parts have to be made precisely at the upper or lower limit of the tolerance range. A part with several contributing dimensions would have to be produced with several possible combinations of dimensions. The parts would probably have to be made using more accurate processes. This is both expensive and time-consuming. The number of experiments increases significantly with the number of variables being studied. Therefore, practical applications would require limiting the number of variables, and thus limit the overall accuracy of the method. There are ways of limiting the number of samples required for an experimental design, but the designs become complex, and determining the effects of the interactions becomes complicated.

Performing the experiments by simulation requires that a mathematical model for the variation be determined. As was discussed earlier, explicit or implicit equations can be used. But implicit equations require an iterative solving technique, increasing the

computation time. Using a computer to perform the simulations permits comprehensive models including many sources of variation. But, the analysis can only be as good as the mathematical model describing the assembly.

Both physical experiments and computer simulation have value in variation analysis. Determining which method to use depends on whether the sources of variation for the assembly are known, and defined by a mathematical model. In fact, if a source of variation is not well defined, an experimental design or response surface could be used to create a mathematical model for it to be included with the other sources of variation.

Section 11.2. Accuracy versus Method and Mixing Designs

In Section 6.7, the accuracy of the different experimental designs in predicting the output distribution moments were presented as percent error from the benchmark of Monte Carlo at one billion samples. And in Section 8.1 the measures of standard moment error (SER) were presented as a better estimate of error. One of the main reasons that SER is better than using percent error is that all of the moment errors are non-dimensionalized by the appropriate power of the standard deviation. If the actual skewness were close to zero, then any error in skewness would cause the percent error to be magnified disproportionately. Table 11-1 below shows the standard moment errors for the five experimental designs investigated.

Table 11-1: Standard Moment Errors versus DOE Method for the Clutch Assembly

Contact Angle	2-Level	2-Level Orthogonal	3-Level	3-Level Orthogonal	3-Level Weighted
SER1.....	0.00004	-0.00018	0.00004	-0.00009	0.00002
SER2.....	-0.00168	0.01223	-0.00126	0.00685	0.00001
SER3.....	0.06529	-0.37446	0.04911	-0.20469	0.00039
SER4.....	-1.40934	-1.34504	-1.06131	-1.35032	-0.01148

The grayed portion of the table shows where significant errors are introduced because those experimental designs model the input variable distributions as uniform instead of

normal. For the clutch assembly problem, all of the methods calculate the mean of for the contact angle well. The orthogonal designs do not estimate the variance as well as the others. The only design to estimate all of the moments well is the three-level weighted design.

The three-level weighted design estimates all four of the moments quite well. But, the weighted design requires all of the runs be performed (a full 3^n factorial). Because of this it is not very practical for problems with many variables. In the case of the clutch assembly, the three-level weighted design analyzes the assembly quite accurately. Table 11-2 below shows how many samples for a Monte Carlo simulation would be required to reach a similar level of accuracy for each of the moments (equating the one-sigma bound on the error with the actual error from the experimental designs).

Table 11-2: Equivalent Monte Carlo Sample Size for Accuracy of DOE Methods on the Clutch Assembly

Contact Angle	2-Level	2-Level Orthogonal	3-Level	3-Level Orthogonal	3-Level Weighted
DOE Runs.....	**8**	**4**	**27**	**9**	**27**
SER1.....	5.6E+08	3.2E+07	7.8E+08	1.2E+08	4.0E+09
SER2.....	7.1E+05	1.3E+04	1.3E+06	4.3E+04	2.0E+10
SER3.....	940	31	1,660	97	2.7E+07
SER4.....	56	61	95	61	7.6E+05

The grayed out portion of the table represents where the analysis methods model the input variables as uniform distributions instead of normal. The equivalent sample sizes that are greater than one billion (1.0E+09) can not trusted, as the Monte Carlo benchmark is only one billion samples large. These approximations are based on Table 11-1, and Eq. 9-1 through Eq. 9-4 (SER versus sample size).

The accuracy of the experimental designs in the case of the clutch assembly is unusually great, probably because the assembly is not very non-quadratic. In fact the results for the weighted three-level design are very similar to using a response surface (sampling at

1.4σ) to obtain sensitivities, and then using the method of system moments to estimate the moments (see Figure 10-1 and Table 10-1). Design of experiments and response surface methods are closely related.

Estimating the expected error using DOE is quite difficult. The error depends on how well the chosen design (two or three level, orthogonal, etc.) can model the actual assembly function. Determining the quadratic ratio (see Section 10.2.1) for the variables that are suspected to have large quadratic effects or large variations can help in designing the experiment.

Before designing and running the experiments, the quadratic ratios should be determined for the variables with the largest variations, or the ones suspected to have large quadratic effects. If the quadratic ratios are large relative to the desired analysis accuracy, those variables should have three levels instead of two. When the variables in a design have different numbers of levels, this is called a mixed design. Two levels can only estimate the linear effects of a variable, and the interactions. Three levels can estimate the quadratic effects for a variable. Therefore, the variables with relatively large quadratic ratios relative to the desired analysis accuracy should have three levels, and the others only two.

In the example of the clutch assembly, the quadratic ratios for the input variables were estimated, along with the estimated and actual linearization error (see Table 10-1 and Table 10-3). Because the hub radius (*a*) has the largest standard deviation of the input distributions, and because it has a large quadratic ratio, using three levels for it should increase the accuracy of the analysis more than using three levels for any of the other variables alone. Table 11-3 below shows the setup and the execution of the mixed experimental design.

Table 11-3: Setup and Response for Mixed Design

	Levels			Values			
Run	**a**	**c**	**e**	**a**	**c**	**e**	**ϕ**
1	-1	-1	-1	27.6246	11.4267	50.7958	**7.28628**
2	-1	-1	+1	27.6246	11.4267	50.8042	**7.38051**
3	-1	+1	-1	27.6246	11.4333	50.7958	**7.13225**
4	-1	+1	+1	27.6246	11.4333	50.8042	**7.22853**
5	+1	-1	-1	27.6654	11.4267	50.7958	**6.80179**
6	+1	-1	+1	27.6654	11.4267	50.8042	**6.90270**
7	+1	+1	-1	27.6654	11.4333	50.7958	**6.63650**
8	+1	+1	+1	27.6654	11.4333	50.8042	**6.73993**
9	0	-1	-1	27.6450	11.4267	50.7958	**7.04818**
10	0	-1	+1	27.6450	11.4267	50.8042	**7.14558**
11	0	+1	-1	27.6450	11.4333	50.7958	**6.88881**
12	0	+1	+1	27.6450	11.4333	50.8042	**6.98848**

The mixed design requires only four more runs than the full two-level design (the grayed portion of the table). The levels for the two, two-level variables are plus and minus one standard deviation. The levels for the three-level variable are the mean and $\pm\sqrt{3/2}\sigma$. The mixed design above is very similar to the two level design performed in Section 6.4 (see Table 6-2), only the mixed design has four extra runs.

The calculations of the main effects and contributions to variance for a mixed design are the same as shown earlier, and so will not be shown here. The output distribution moments from the mixed design are very similar to the three-level design. The standard moment errors for the two and three-level and mixed designs are shown below in Table 11-4.

Table 11-4: Relative Error Comparison for Mixed Design

Contact Angle	2-Level	Mixed	3-Level
SER1.....	0.000042	0.000036	0.000036
SER2.....	-0.00168	-0.00127	-0.00126
SER3.....	0.0653	0.0496	0.0491
SER4.....	-1.4093	-1.0714	-1.0613
Number of Runs.....	**8**	**12**	**27**

The standard moment errors for the mixed design are very similar to the three-level design. About 95 percent of the difference between the two and three-level designs is eliminated by adding just four extra runs. The mixed design still has less than half of the runs of the full three-level design. If the experiments were being performed by physically building the assemblies, maximizing the accuracy while minimizing the number of runs would really save time and money. Even orthogonal designs can be mixed. Thus, estimating the quadratic ratio for the key variables can really help optimize the analysis.

Section 11.3. Efficiency of Response Surface Designs to Estimate Sensitivities

The finite difference method of obtaining sensitivities for a full quadratic model was investigated by Glancy [Glancy 1994, 24]. Eq. 11-1 below was determined to be the number of samples required versus the number of variables.

$$\#Runs = 2n^2 + 1 \qquad \textbf{Eq. 11-1}$$

For the central composite design (CCD), shown graphically in Figure 6-1, the required number of runs versus the number of variables is determined by Eq. 11-2 below. The term *c* is the number of runs done at the center (all of the variables at their mean values). Multiple points are often run at the center to get an idea of the pure experimental variation. When using a mathematical model this variation is most often very small, therefore only one point is required at the center.

$$\#Runs = 2^n + 2n + c \ \text{(multiple center runs)} \qquad \#Runs = 2^n + 2n + 1 \ \text{(single center run)} \qquad \textbf{Eq. 11-2}$$

For the Box-Behnken response surface design (BOX), shown graphically in Figure 6-1, the number of runs is given by the Eq. 11-3 below. Again the number of center points can be limited to just one.

$$\#Runs = 2n(n-1) + c \ \text{(multiple center runs)} \qquad \#Runs = 2n(n-1) + 1 \ \text{(single center run)} \qquad \textbf{Eq. 11-3}$$

For both of these response surface designs, the number of required runs can be reduced. A half-factorial or less can replace the full (2p) factorial portion of the CCD. However, the ability to detect interactions may be reduced. The Box-Behnken design can likewise be made more efficient for higher numbers of variables [Myers 1995].

In order to fit a full quadratic model, the number of coefficients (or factors) in the quadratic equation is the same as the number of sensitivities needed, plus the constant term. The number of factors is given by Eq. 11-4.

$$\begin{aligned} \#Factors &= (\text{constant}) + (\text{first - order terms}) \\ &\quad + (\text{second - order terms}) + (\text{interaction terms}) \\ &= 1 + n + n + \frac{n(n-1)}{2} \\ &= 1 + 2n + \frac{n(n-1)}{2} \end{aligned} \qquad \textbf{Eq. 11-4}$$

From the number of factors in the quadratic model, the efficiency of the method for calculating the sensitivities will be defined as the number of factors divided by the number of runs required to find the sensitivities. Table 11-1 shows three methods to calculate sensitivities, the number of factors, and the efficiency for several different numbers of variables.

Table 11-1: Method to Find Sensitivities versus the Number of Runs

		Number of Runs			Efficiency (Factors/Runs)		
Variables	Factors	Finite Diff.	CCD	BOX	Finite Diff.	CCD	BOX
1	3	3	5	NA	100.0%	60.0%	NA
2	6	9	9	NA	66.7%	66.7%	NA
3	10	19	15	13	52.6%	66.7%	76.9%
4	15	33	25	25	45.5%	60.0%	60.0%
5	21	51	43	41	41.2%	48.8%	51.2%
6	28	73	77	61	38.4%	36.4%	45.9%
7	36	99	143	85	36.4%	25.2%	42.4%
8	45	129	273	113	34.9%	16.5%	39.8%
9	55	163	531	145	33.7%	10.4%	37.9%
10	66	201	1,045	181	32.8%	6.3%	36.5%

The finite difference method is very similar in efficiency to the Box-Behnken design. The number of runs required for the central composite design increases dramatically, but is very efficient for 2–4 variables. Both of the response surface designs can be modified to be more efficient with larger numbers of variables, but the designs get more complex. The efficiency information in the table above is more easily seen in Figure 11-1 below. The Box-Behnken design is not defined for only one or two variables, so those points do not show up in the figure.

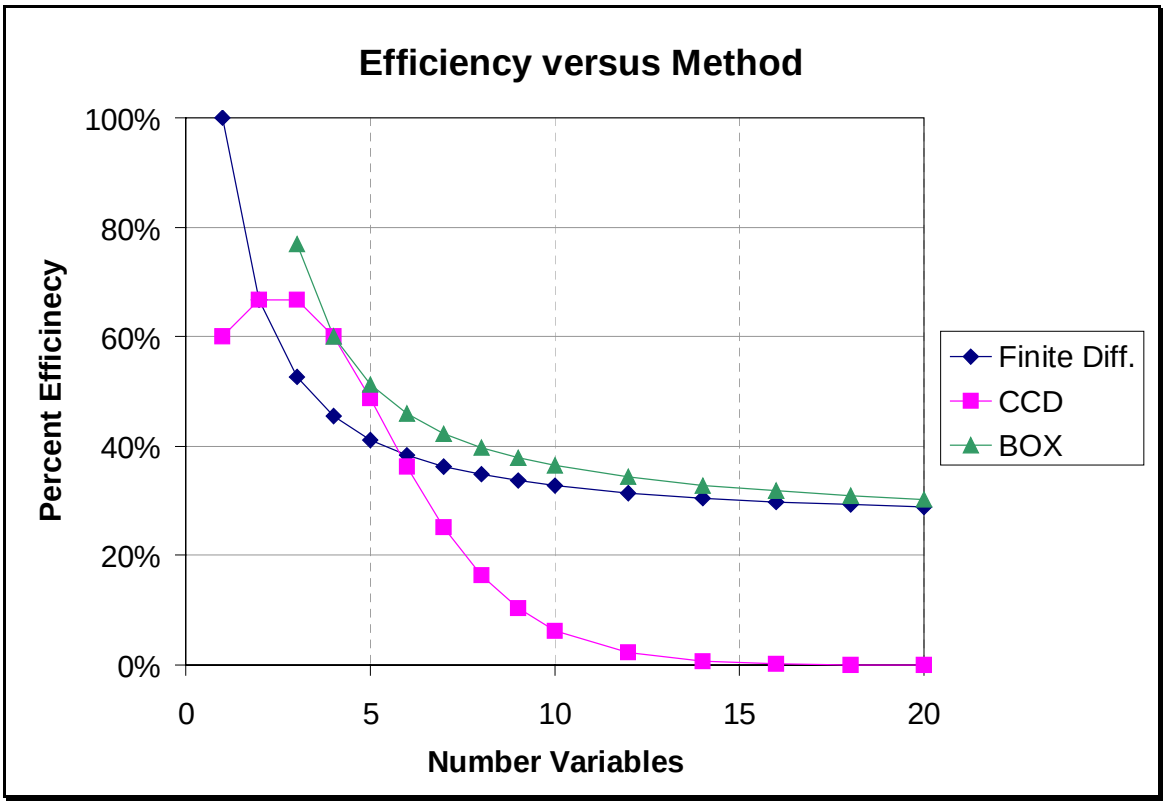


Figure 11-1: Method Efficiency for Calculating Sensitivities

The central composite design is very competitive with the other methods at low numbers of variables, but becomes inefficient very quickly. The Box-Behnken design is a little bit more efficient than the finite difference method at calculating sensitivities.

From the several different methods to calculate the sensitivities, it is clear that sampling the response function at points between one and two standard deviations away from the means of the input variables gives the best results for the method of system moments. Chapter 12 will discuss the reasons for this increase in accuracy, and will present even another method or estimating the sensitivities.

Section 11.4. Summary for Estimating the Effectiveness of DOE and Response Surface Methods

Design of experiments can be very accurate in estimating the output distribution moments. Although it is difficult to predict the accuracy of DOE, using the quadratic ratio can help determine which variables might need three levels. Generally no more than three levels for an input variable is used, especially in variation analysis where the variations are relatively small.

Design of experiments can not easily model non-normal input distributions. The regular two and three-level designs model the input variables as uniform distributions. The mean and variance of the output distribution can be estimated relatively well by any of the experimental designs. For this reason they are well matched with the Taguchi quality loss function, which is only a function of the mean and variance. If the input distributions are normal, the weighted, three-level design can be used to estimate the skewness and kurtosis of the output distribution.

The response surface method is very useful for obtaining sensitivities and is competitive with the finite-difference method. But with large numbers of variables, the sample size for the response surface designs either becomes large, or the designs complicated. The next chapter will investigate how using random points to determine the response surface and sensitivities for MSM can work together well as a hybrid method.

Chapter 12. Random Response Surface Hybrid Method

The methods discussed up to this point (Monte Carlo, method of system moments, RSS linear analysis, and design of experiments) all have different strengths and weaknesses. Additionally, the methods do not all require the same input information, nor do they all provide the same information about the output distribution. Because of these differences, the methods may sometimes be used together to increase their accuracy and efficiency.

It was shown in Section 10.1.1 that using a response surface to estimate the sensitivities can significantly increase the accuracy of the method of system moments. This is a good match as the method of system moments requires the sensitivities to determine the output distribution moments, and a response surface produces a quadratic estimate of the assembly function from which the sensitivities can readily be obtained. But why would the sensitivities from the response surface be better than those obtained by a finite difference method, or from determining the exact derivatives from the explicit assembly function?

Section 12.1. Optimal Sensitivities for the Method of System Moments

In order to determine how optimum sensitivities for the method of system moments can be obtained, the methods already discussed will be compared graphically. The function for which the sensitivities are to be estimated is one for which the non-quadratic effect is exaggerated. Eq. 12-1 below shows this function, which is only of one variable to allow for easy comprehension.

$$Y = (X + 1)^4 \qquad \textbf{Eq. 12-1}$$

A linear approximation for a first order method of system moments (or RSS linear analysis) requires just the first derivative of the function with respect to the input variable. A second-order approximation would require the second derivative. Because the function is simple the derivative can easily be determined explicitly. A finite-difference method

would obtain very similar sensitivities. These sensitivities will be referred to as Taylor sensitivities, because they approximate the function as a polynomial (linear in this case) for which the accuracy is exact at the nominal, and gets worse as the distance from the nominal increases.

For this demonstration the variable X is defined as a standard normal variable (mean of zero, and standard deviation of 1.0). The calculation of the first and second derivatives of the function at the mean of X are shown below in Eq. 12-2 and Eq. 12-3.

$$\frac{\partial Y}{\partial X} = 4(X+1)^3 = 4 \qquad \textbf{Eq. 12-2}$$

$$\frac{\partial^2 Y}{\partial X^2} = 12(X+1)^2 = 12 \qquad \textbf{Eq. 12-3}$$

Using the derivatives above, and the Taylor series expansion, the linear and the quadratic approximations to the original function are shown below in Eq. 12-4 and Eq. 12-5.

$$Y_{Linear,Taylor} = 4X + 1 \qquad \textbf{Eq. 12-4}$$

$$Y_{Quadratic,Taylor} = 6X^2 + 4X + 1 \qquad \textbf{Eq. 12-5}$$

Another method to obtain the approximation for the original function (or the sensitivities) is design of experiments. A two-level experimental design would be like fitting a line between the true function values at X = -1 and +1 (one standard deviation). Similarly, the three-level experimental design would be like fitting a quadratic function to the three true function values at X = -1.225, 0.0, and 1.225 ($-\sqrt{3/2}\sigma$, 0, and $+\sqrt{3/2}\sigma$). Using the true values of the function at these points, and fitting a line and a quadratic yields Eq. 12-6 and Eq. 12-7.

$$Y_{Linear,2-Level} = 8X + 8 \qquad \textbf{Eq. 12-6}$$

$$Y_{Quadratic,3-Level} = 7.5X^2 + 10X + 1 \qquad \textbf{Eq. 12-7}$$

The original function, the linear Taylor approximation, and the two-level approximation are shown below in Figure 12-1 for comparison.

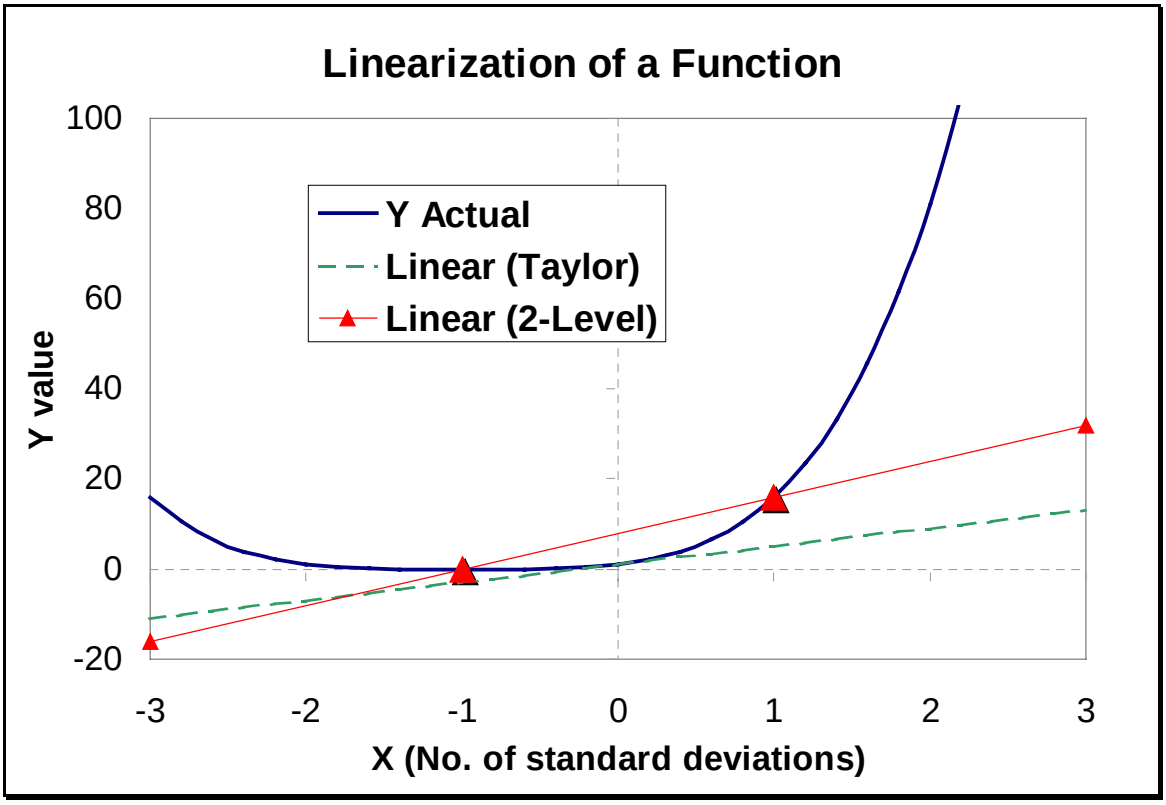


Figure 12-1: Conventional Methods to Linearize a Function

The Taylor linear approximation is only a good approximation of the actual function right at the nominal value of X (0.0). The two-level approximation fits the function reasonably well between –2 and about 1.5. The big triangles represent the values used to create the two-level linear approximation.

The two linear approximations are not designed to get the same approximation of the output function. The Taylor approximation is designed to have the same value and slope of the actual function at the nominal value of *X*. The two-level approximation is designed to get a better linear approximation of the function over the range of probability. The slope of the two-level approximation is two times the slope of the Taylor approximation;

thus estimated variance of the output function would be four times greater (closer to the variation of the actual function). This is one of the reasons that a two-level DOE is generally more accurate than linear analysis with the partial derivatives (RSS). The two quadratic approximations for the actual function are shown together below in Figure 12-2.

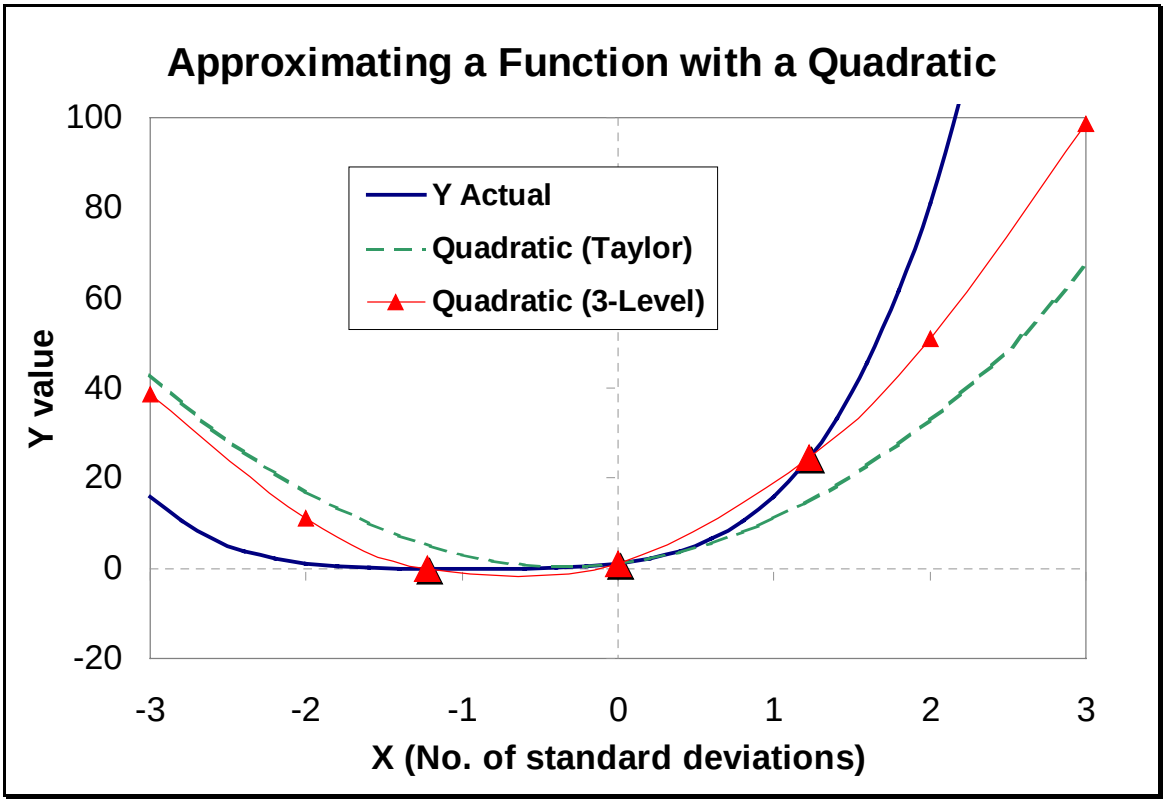


Figure 12-2: Conventional Methods to Approximate a Function by a Quadratic

The Taylor approximation is exact right around the nominal value of X (0.0), but does not approximate the original function well farther away. The three-level approximation fits the true curve better all over, but not the slope at $X = 0.0$. The three-level approximation does not give a good estimate for the exact first and second derivative of the function at the nominal value of X, but rather a better estimate of the values over the range the function.

In Section 10.1, various response surfaces were used to approximate the clutch assembly function with a quadratic (very similar to the three-level approximation above). It was shown in that section that sampling the value of the function with the input variables at about 1.4 to 1.8 times their standard deviations obtained much more accurate estimates of the output distribution moments when using the method of system moments. Sampling at only 1.4 times the standard deviations (rather that at two or three) takes into account the fact that more values of X will be clustered near the mean than in the tails of the normal distribution.

Thus, using the method of system moments with input from an experimental design or a response surface can enhance its accuracy. But, the most accurate method to determine the sensitivities for MSM would be to fit a response surface to many points over the range of the function weighted by their probability of occurring. An easier method would be to use Monte Carlo to generate the random points according probability distributions for the input variables. More points would be generated around the mean, and fewer at the tails, and there would be no need to weigh the points. This randomly created response surface will be called a random response surface. Figure 12-3 below shows the results of a linear and quadratic random response surface using 100 samples of Monte Carlo.

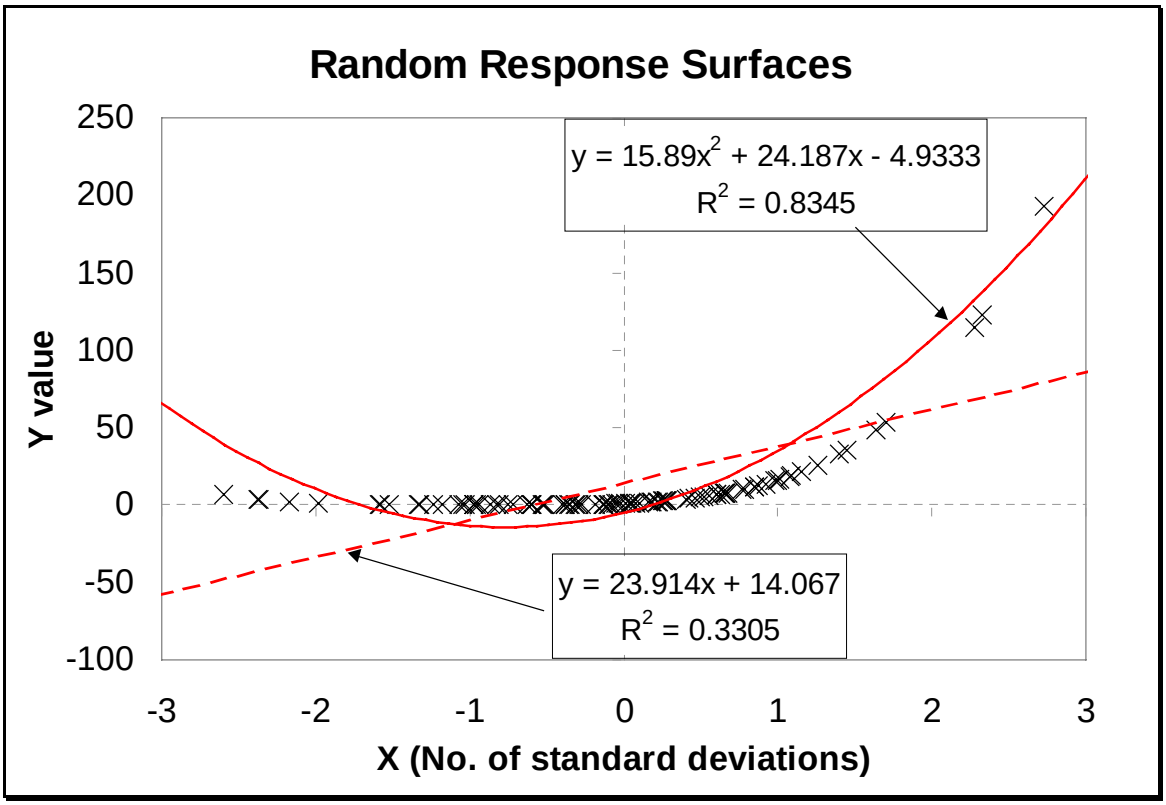


Figure 12-3: Linear and Quadratic Random Response Surfaces

The points shown on the graph as an "x" are the 100 randomly generated function values. The linear and quadratic regression curves are shown along with their equations. The R^2 term, or coefficient of determination, represents the fraction of the total variation in the response accounted for by the fit equation, 83 percent for the quadratic case [Vardeman 1994, 107]. The coefficient of determination is calculated using Eq. 12-8 below.

$$R^2 = \frac{\sum_{i=1}^{n}\left(Y_i-\bar{Y}\right)^2 - \sum_{i=1}^{n}\left(f(X_i)-Y_i\right)^2}{\sum_{i=1}^{n}\left(Y_i-\bar{Y}\right)^2} = \frac{\text{Variation of the fit equation}}{\text{Variation of the raw data}} \qquad \textbf{Eq. 12-8}$$

Where: Y_i = The i^{th} actual response

$\bar{Y}$ = The mean of the actual responses

$f(X_i)$ = The value of the fitted equation for the same input variable values as the i^{th} actual response

The coefficient of determination is a valuable tool in estimating the effectiveness of the fit equation to model the variance of the original function. The accuracy of the random response surface now is a function of the sample size, and how well a quadratic function can indeed model the true function. The random response surface is the hybrid analysis method that will be the focus of the rest of this chapter.

Section 12.2. Random Response Surfaces for MSM Sensitivities

The experimental designs used thus far to get the sensitivities are mostly symmetrical, and good for normally or uniformly distributed input variables. A more general method devised through this thesis to determine a good response surface is to use Monte Carlo Simulation to generate the points to fit a response surface. The random response surface would then truly minimize the error in modeling a real function with a quadratic and could account for non-normal inputs.

One method to determine a quadratic equation that approximates the real function from the random function evaluations is to minimize the sum of the squares of the residuals of the fit. This is the most popular method, and the one used in the previous section for the example. Eq. 12-1 below represents the error function to minimize.

$$\text{Sum - Squared Error} = \sum_{i=1}^{n} \left(Y_i - f(X_i)\right)^2 \qquad \textbf{Eq. 12-1}$$

Where: Y_i = The real value of the assembly function at the i^{th} values of the input variables

$f(X_i)$ = The fitted equation (quadratic in this case) evaluated at the i^{th} values of the input variables

To minimize the value of Eq. 12-1, taking the partial derivatives with respect to the coefficients for the fitted equation yields a simple solution by linear algebra. Because of the simple solution to this “least-squares fit,” this method of curve fitting is extremely popular. A simplified derivation for the linear algebra solution will be presented here to show that the mean of the of the actual function observations end up equal to the mean of

the fitted equation for the data points. The equation for sum-squared error is shown below in Eq. 12-2 with the fit equation expanded out. The *m* functions shown below depend on the type of fit being performed (linear, quadratic, etc.)

$$Error = \sum_{i=1}^{n} \left[Y_i - \left(C_0 + C_1 f_1(X_i) + C_2 f_2(X_i) + \cdots + C_m f_m(X_i) \right) \right]^2 \qquad \textbf{Eq. 12-2}$$

To determine the coefficients of the fit equation to minimize the sum-squared error, the partial derivatives with respect to the coefficients are taken as shown below in Eq. 12-3.

$$\begin{aligned}
\frac{\partial Error}{\partial C_0} &= 0 = 2 \sum_{i=1}^{n} \left\{ \left[Y_i - \left(C_0 + C_1 f_1(X_i) + C_2 f_2(X_i) + \cdots + C_m f_m(X_i) \right) \right] (-1) \right\} \qquad \textbf{Eq. 12-3} \\
\frac{\partial Error}{\partial C_1} &= 0 = 2 \sum_{i=1}^{n} \left\{ \left[Y_i - \left(C_0 + C_1 f_1(X_i) + C_2 f_2(X_i) + \cdots + C_m f_m(X_i) \right) \right] (-f_1(X_i)) \right\} \\
\frac{\partial Error}{\partial C_2} &= 0 = 2 \sum_{i=1}^{n} \left\{ \left[Y_i - \left(C_0 + C_1 f_1(X_i) + C_2 f_2(X_i) + \cdots + C_m f_m(X_i) \right) \right] (-f_2(X_i)) \right\} \\
&\vdots \\
\frac{\partial Error}{\partial C_m} &= 0 = 2 \sum_{i=1}^{n} \left\{ \left[Y_i - \left(C_0 + C_1 f_1(X_i) + C_2 f_2(X_i) + \cdots + C_m f_m(X_i) \right) \right] (-f_m(X_i)) \right\}
\end{aligned}$$

The set of equations above can then be easily transformed into a matrix algebra problem as shown below in Eq. 12-4.

$$\begin{bmatrix} Y_i \\ Y_i f_1(X_i) \\ \vdots \\ Y_i f_m(X_i) \end{bmatrix} = \begin{bmatrix} n & f_1(X_i) & \cdots & f_m(X_i) \\ f_1(X_i) & (f_1(X_i))^2 & \cdots & f_1(X_i) f_m(X_i) \\ \vdots & \vdots & \vdots & \vdots \\ f_m(X_i) & f_m(X_i) f_1(X_i) & \cdots & (f_m(X_i))^2 \end{bmatrix} \begin{bmatrix} C_0 \\ C_1 \\ \vdots \\ C_m \end{bmatrix} \qquad \textbf{Eq. 12-4}$$

Solving the linear algebra expression above solves for the coefficients that solve Eq. 12-3 and minimizes the sum-squared error (Eq. 12-1). The first equation of Eq. 12-3 can be simplified, as shown below in Eq. 12-5.

$$0 = 2 \sum_{i=1}^{n} \{[Y_i - (C_0 + C_1 f_1(X_i) + C_2 f_2(X_i) + \cdots + C_m f_m(X_i))](-1)\} \quad \textbf{Eq. 12-5}$$

$$0 = \sum_{i=1}^{n} Y_i - \sum_{i=1}^{n} (C_0 + C_1 f_1(X_i) + C_2 f_2(X_i) + \cdots + C_m f_m(X_i))$$

$$0 = \sum_{i=1}^{n} Y_i - \sum_{i=1}^{n} f(X_i)$$

The equation above shows that for the least-squares fit surface, the sum of the original function values from the data is equal to the sum of the fitted values from the data. Thus the means will also be equal, except for the round-off error. Thus SER1 can not be predicted with the same data points used to fit the response surface.

Minimizing the sum-squared error (Eq. 12-1) is almost like minimizing the error in variance between the true equation and the fitted approximation (SER2). Figure 12-1 below shows how the random response surface method performs using the least-squares method to fit the random points.

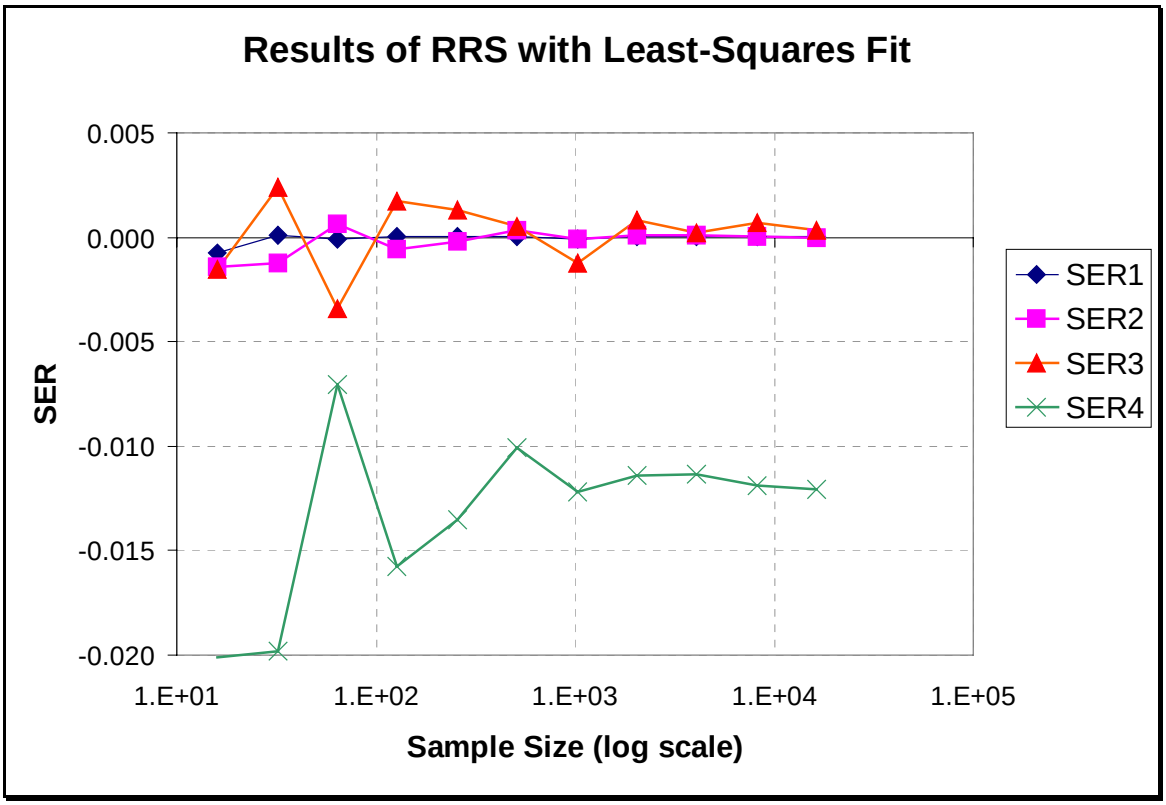


Figure 12-1: Random Response Surface versus Sample Size using Least-Squares Fit

The standard moment errors generally decrease with sample size. But, after a few hundred samples, the standard moment errors do not decrease much. The standard moment error for kurtosis (SER4) is much larger relative to the others. The points of the graph represent one run (all of them starting from the same random seed value).

Another method to fit a function is to minimize the fourth power of the residuals. The error function to minimize is very similar to the sum-squared error function presented above. Eq. 12-6 below shows the sum-fourths error function.

$$\text{Sum - Fourths Error} = \sum_{i=1}^{n} \left(Y_i - f(X_i)\right)^4 \qquad \textbf{Eq. 12-6}$$

The least-fourths method to fit data (minimizing the sum-fourths error function) is not as easy as the sum-squares method. There is no simple linear algebra solution. Eq. 12-6 must

be minimized by the method of steepest decent or some other minimization algorithm. Minimization algorithms are not guaranteed to find the global optimum, and require much more time to solve. Using the method of steepest decent, and a starting point for the equation coefficients from the least-squares fit, comparative results to Figure 12-1 were obtained. Figure 12-2 below shows how the output distribution moments calculated from the sensitivities obtained from random points and a least-fourths response surface fit.

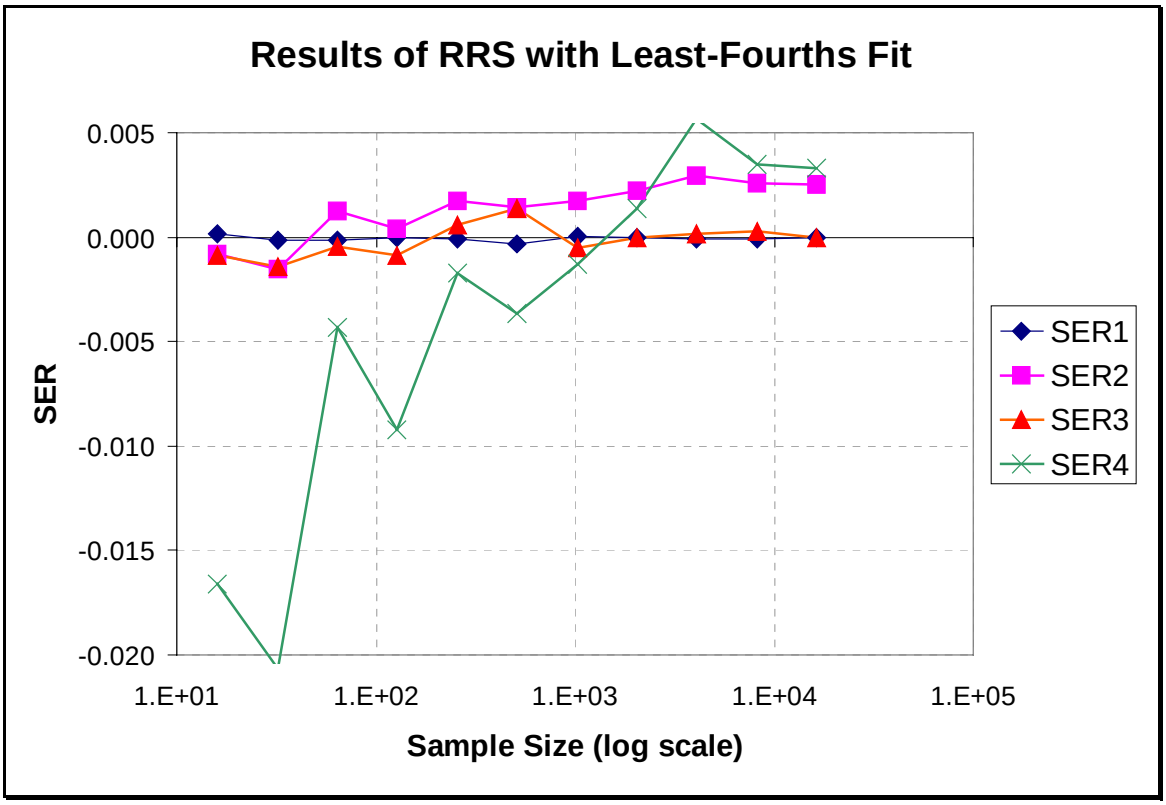


Figure 12-2: Random Response Surface versus Sample Size using Least-Fourths Fit

Minimizing the sum-fourths error does appear to reduce the error in kurtosis (SER4). The minimization algorithm did not necessarily find the optimum response surface for the samples. While the error in kurtosis decreased, the error in variance increased. Starting the optimization for the least-fourths method at the solution to the least-squares method seems to be successful.

Both the least-squares and the least-fourths methods could be used together. The least-squares method could be used to determine the mean, variance, and possible skewness. The least-fourths method could then be used to estimate the kurtosis. But this would generate two equations instead of just one to model the function. Additionally, using both methods together would be complicated and time consuming.

The rest of the chapter will use the least-squares method of response surface fitting, as the objective function (PPM rejects and quality loss) are generally more sensitive to errors in variance than kurtosis. This random response method (RRS) seems to obtain a good estimation of the output distribution moments with relatively few samples compared to Monte Carlo simulation. The next section will try to determine the accuracy of the RRS method.

Section 12.3. Estimating the Error of the RRS Method

The random response surface method (RRS) used randomly generated function evaluations (according to the probability distributions of the input variables) to find a quadratic response surface for the method of system moments. The first four output distribution moments from the method of system moments could then be used to estimate the PPM rejects and the quality loss. The main errors involved in the RRS method come from the following:

- Input information error (input variable distributions, unknown sources of variance, etc.)
- Modeling a real function as a quadratic (fitted to minimize the error in variance)
- Using random numbers to determine the data for the response surface fit

All of these sources of error affect the accuracy of the analysis method. But, as with the other methods, errors in the input information (input distributions, specification limits, etc.) can not be overcome by increasing the accuracy of the analysis. Therefore, the

inherent accuracy of the RRS method should be determined to understand when it is and is not appropriate to use.

12.3.1. Estimating the Standard Moment Errors for the Fit

The first source of error that will be estimated is the error of trying to use a quadratic equation to model a real assembly function. As in the case of linear analysis, if the variations of the input variables are small relative to the higher order sensitivities (third-order and greater) then a quadratic approximation will be good. Section 10.1.2 presented the Monte Carlo Difference method (MCD) to help determine the accuracy of MSM with much fewer samples than Monte Carlo alone. The same method (MCD) will be used to estimate the accuracy of the random response surface method.

After the Monte Carlo random points are generated and the original assembly function evaluated at the points, the response surface can be fit to the data. The previous section described how a least-squares fit is used to minimize the squared error of the fit.

The R^2 coefficient (or the coefficient of determination) was shown to be a measure of the percentage of the total variation in the data that is accounted for by the fit equation. Although the coefficient of determination is very useful for the error of variance, and widely used, the measures of fit that will be used in this paper are the estimates of the standard moment error. Eq. 12-1 below shows the formula for the actual standard moment errors and the approximate errors using the MCD method.

$$SERj = \frac{\hat{\mu}_j - \mu_j}{\mu_2^{j/2}} \approx \frac{\mu_{j,f(X)} - \mu_{j,Y}}{\mu_{2,Y}^{j/2}} \qquad \textbf{Eq. 12-1}$$

Where: $\mu_{j,f(X)} = \frac{1}{n}\sum_{i=1}^{n}\left(f(X_i) - \mu_{1,f(X)}\right)^j$ = The j^{th} moment for the fit function evaluated at the Monte Carlo generated points

$\mu_{j,Y} = \frac{1}{n}\sum_{i=1}^{n}\left(Y_i - \mu_{1,Y}\right)^j$ = The j[th] moment for the original assembly function evaluated at the Monte Carlo generated points

The reason that the standard moment calculations are only approximations is that the moments from the Monte Carlo generated data are used as the estimates for the true values, and their accuracy is based on the sample size. It is significant that the estimates for SER1–4 give not only the magnitude, but also the sign of the error, unlike the coefficient of determination! Figure 12-1 below shows the actual standard moment error for the mean and the approximations using the MCD method for the clutch assembly.

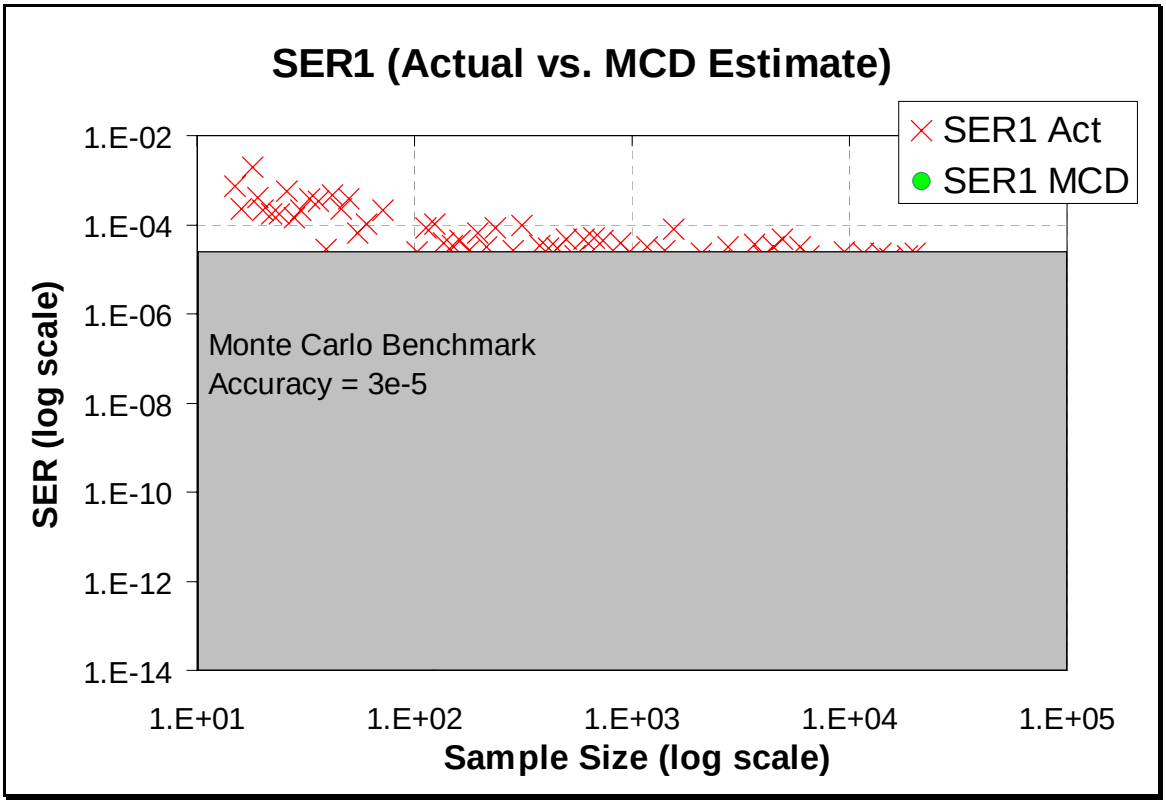


Figure 12-1: SER1 for the RRS Method: Actual Error versus MCD Estimate

The benchmark for the mean is one billion samples of Monte Carlo simulation. The grayed out region represents the limitation of accuracy for using the benchmark to

estimate the actual SER1. The accuracy of the benchmark is only about SER1 = 3e-5, based on the $1/\sqrt{n}$ estimate. But the RRS method is approaching and surpassing that level of accuracy after only one thousand samples. The estimate of SER1 from the MCD method is very small, about $1x10^{-12}$. Eq. 12-3 showed that because the response surface was fit to the same data points used to estimate the standard moment errors, SER1 would be zero, or only close due to rounding error. The figure above confirms that in this case the MCD method can not estimate SER1. Figure 12-2 below shows the same type of graph for the estimate of SER2.

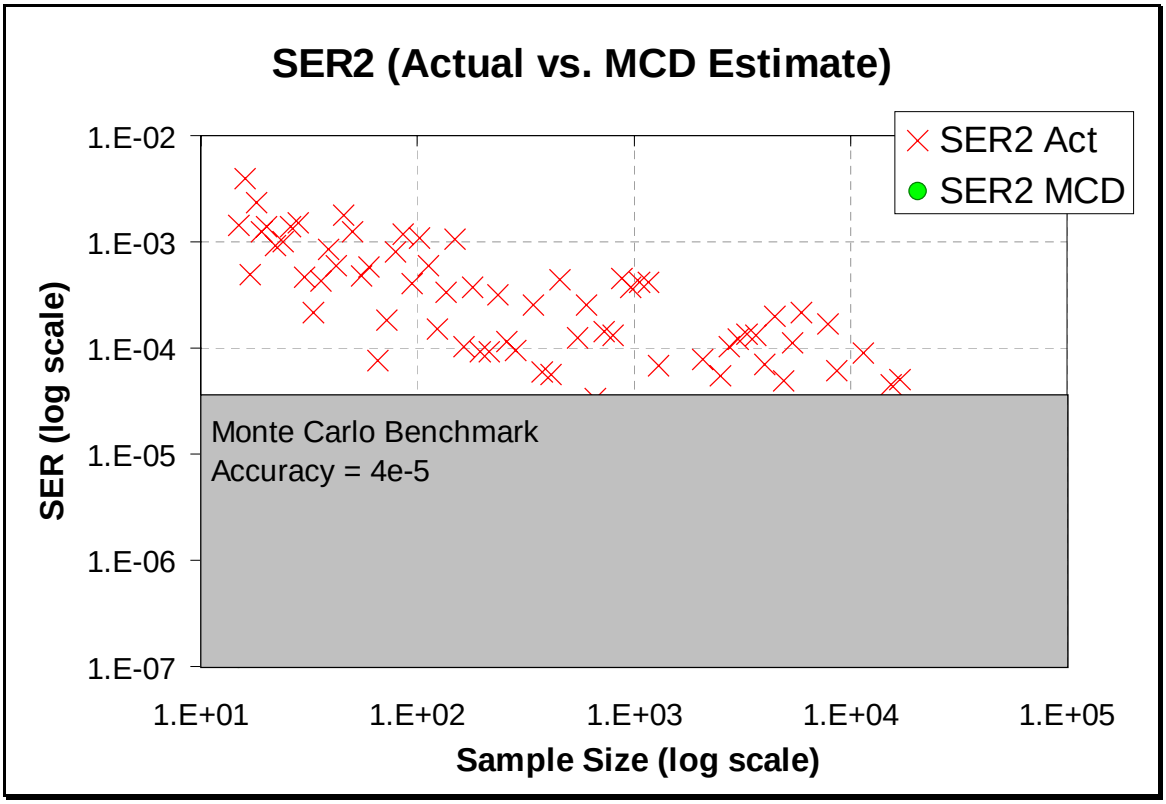


Figure 12-2: SER2 for the RRS Method: Actual Error versus MCD Estimate

Again the accuracy of the Monte Carlo benchmark is surpassed, now at about ten thousand samples. The two estimates of SER2 are starting to converge as the sample size increases. The SER2 calculated from the fit equation is increasing with the sample size, as the SER2 calculated from the Monte Carlo benchmark is decreasing. In theory, if the

benchmark were truly accurate, the two estimates of SER2 would converge to each other. It is not clear whether the MCD estimate of SER2 really is a good estimate (and the Monte Carlo benchmark is bad), or whether (as with SER1) SER2 can not be estimated with the same data as the response surface fit.

The value of the coefficient of determination (R^2) is the percent of the variation that is accounted for by the fit surface, while the estimate of SER2 from the MCD method represents the percent error in variation. Therefore, the quantity $1-R^2$ should be the same as the absolute value of SER2. In the case above, for all of the different sample sizes, the greatest difference between the absolute value of SER2 (estimated with the MSD method) and ($1-R^2$) was 2.0×10^{-10}. The two quantities are essentially equal (except for rounding error).

In Figure 12-3 below, the accuracy of the Monte Carlo benchmark is sufficient to allow the two estimates of SER3 to converge.

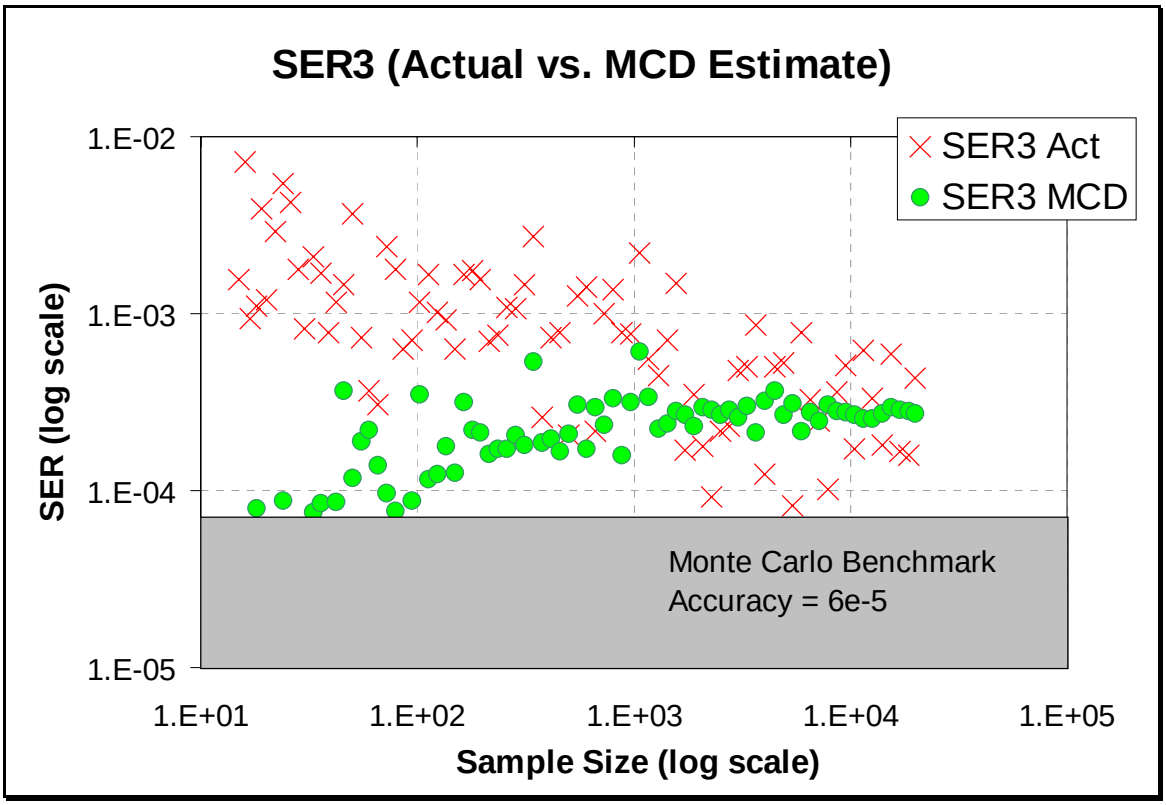


Figure 12-3: SER3 for the RRS Method: Actual Error versus MCD Estimate

The accuracy of the RRS method does not surpass the Monte Carlo benchmark. But the estimate of SER3 from the fit equation is a very good estimate of the actual SER3 calculated from the benchmark after about one thousand samples. The estimate of SER3 from the fit equation is not only a good estimate of the magnitude of SER3 (the graph only shows the absolute value), but it also predicts the sign of SER3. The next section will use the estimate of SER from the fit equation to correct the MSM estimate of the output distribution moments to even further enhance the accuracy of the RRS method.

Figure 12-4 below is very similar to the graph of the estimates of SER3, but it shows even better the convergence of the two estimates of SER4.

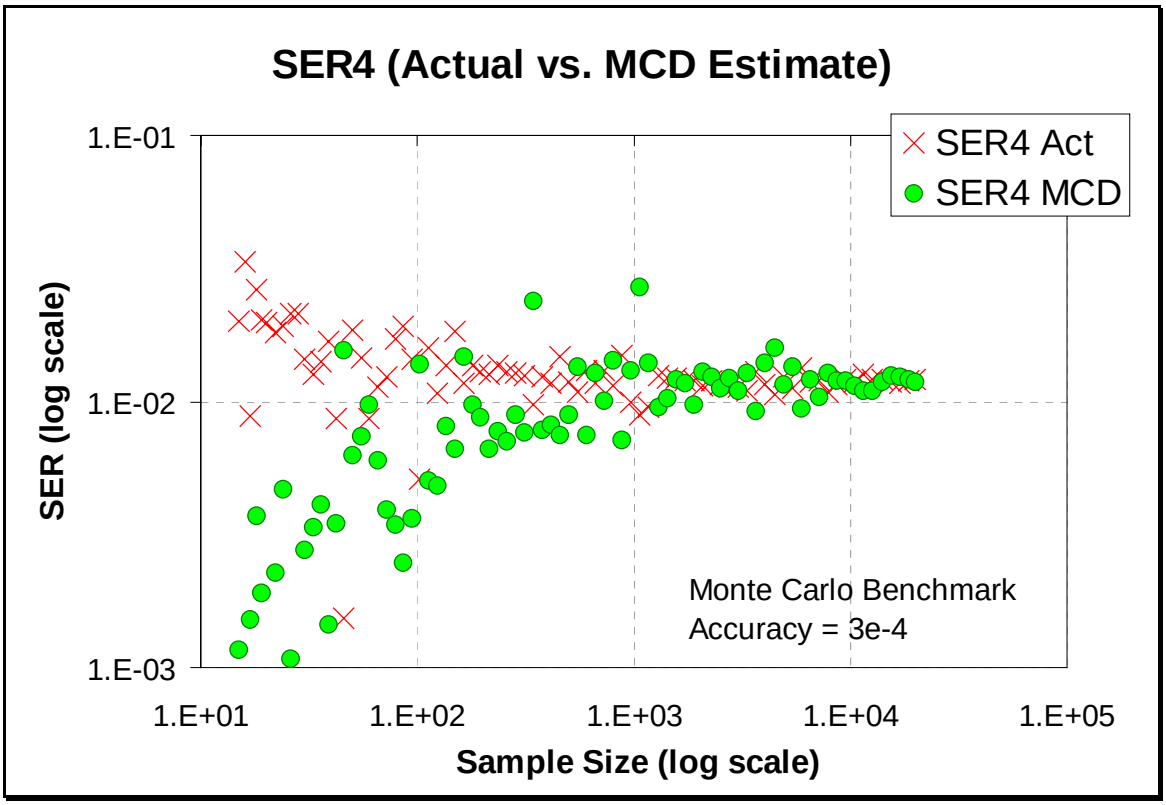


Figure 12-4: SER4 for the RRS Method: Actual Error versus MCD Estimate

The two estimates for SER4 (from the fit equation and from the Monte Carlo benchmark) converge to about the same value of $1x10^{-2}$ after one thousand samples. The region where the Monte Carlo benchmark is not accurate is not even in the range of the graph ($3x10^{-4}$). The estimate of SER4 using the fit equation (see Eq. 12-1) is a good predictor of the true error. Thus, the estimate of SER could be used to adjust the output distribution moments found from the MSM with the fit equation sensitivities.

12.3.2. Adjusting the Moment Estimates

The estimates of SER from the fit quadratic equation can be used to increase the accuracy of the RRS method. But those estimates are a function of the sample size, and the estimates are generally low at smaller sample sizes. To convert the dimensionless standard moment errors to adjustment for the moments themselves, they must be scaled

by the second moment of the distribution to the correct power. This is the reverse of the process in Eq. 12-1. Eq. 12-1 below shows how the estimates of the output distribution moments can be adjusted to increase their accuracy.

$$\mu_{j,adjusted} = \mu_{j,MSM} - SERj\left(\mu_{2,MSM}\right)^{j/2} \qquad \textbf{Eq. 12-1}$$

Where: $\mu_{j,adjusted}$ = The new, more accurate estimate of the j^{th} output moment

$\mu_{j,MSM}$ = The output moment calculated from the sensitivities using MSM

$SERj$ = The estimate of the j^{th} standard moment error from the MCD method

The adjustment represents the error for the moments that could not be predicted by a quadratic model. It is very similar to the error term in the analysis of an experimental design. The mean could not be adjusted because the MCD estimate for the mean errors are essentially zero (as discussed earlier). Figure 12-1 below shows the standard moment error for the mean using the RRS method. The σ_{SER1} using Monte Carlo (see Section 9.2) is also shown for comparison.

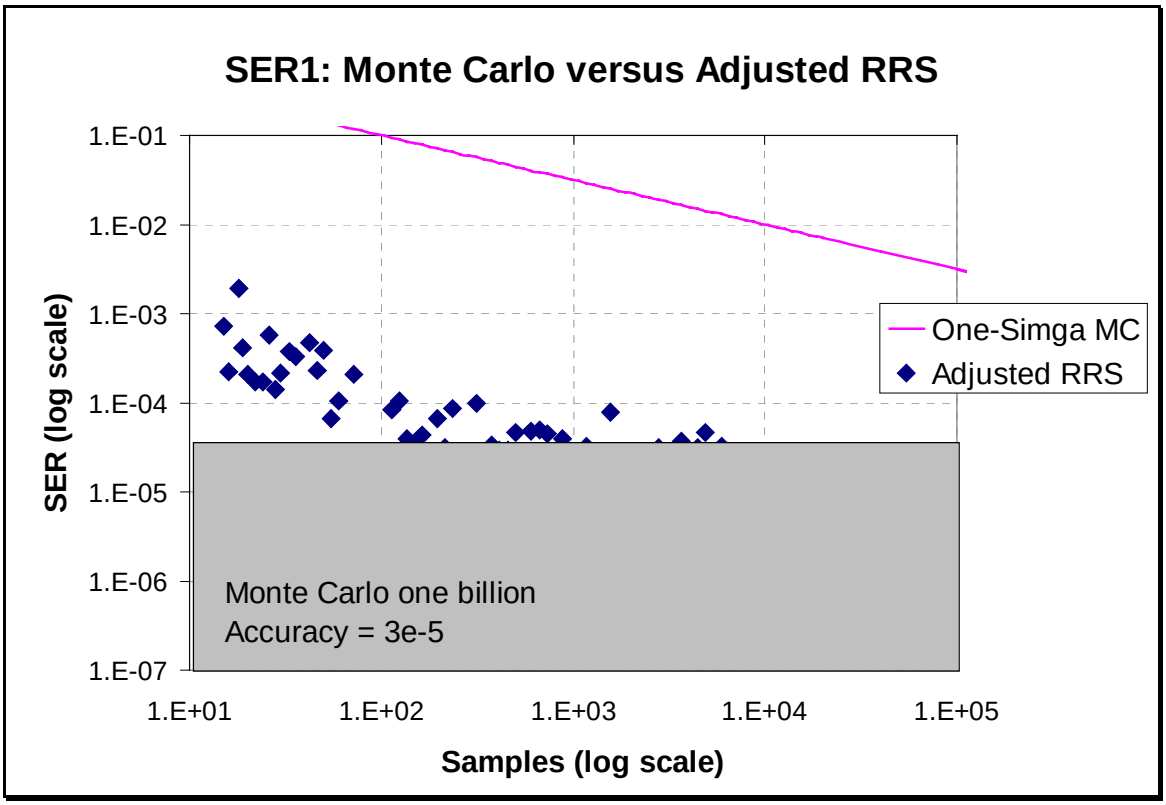


Figure 12-1: Comparing SER1 for RSS and Monte Carlo

The error of the RRS method seems to decrease proportionally to the error of Monte Carlo for the mean. But the adjusted RRS error is much smaller. The same error for one billion Monte Carlo samples can be achieved with only about ten thousand adjusted RRS samples. The accuracy of the adjusted RRS method can not really be evaluated past ten thousand samples because it surpasses the accuracy of the benchmark. The comparisons for SER2 are very similar to SER1 as shown below in Figure 12-2.

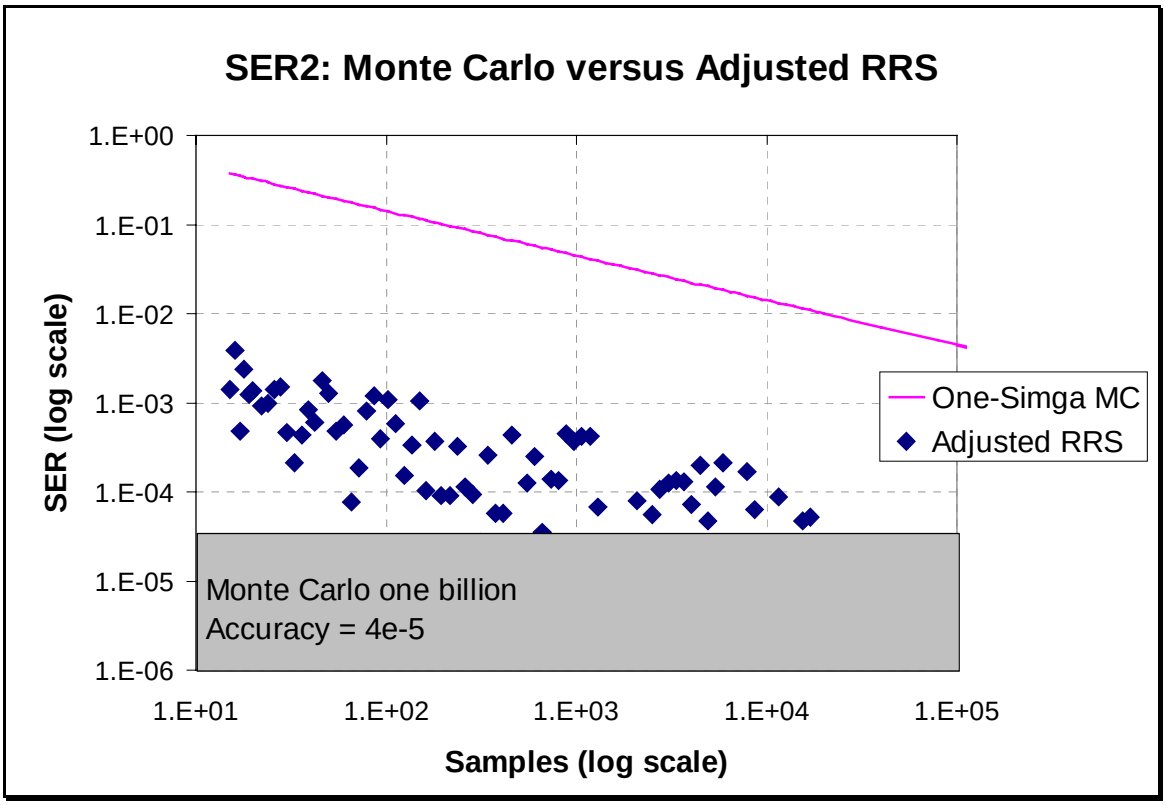


Figure 12-2: Comparing SER2 for Adjusted RSS and Monte Carlo

The accuracy of the adjusted RRS method at ten thousand samples is similar to the accuracy of Monte Carlo at one billion samples. The adjusted RRS method gives extremely accurate estimates of both the mean and variance at low numbers of samples relative to Monte Carlo, a factor of about one hundred thousand fewer samples. The adjusted RRS results look similar to the unadjusted results. The comparisons for SER3 are shown below in Figure 12-3.

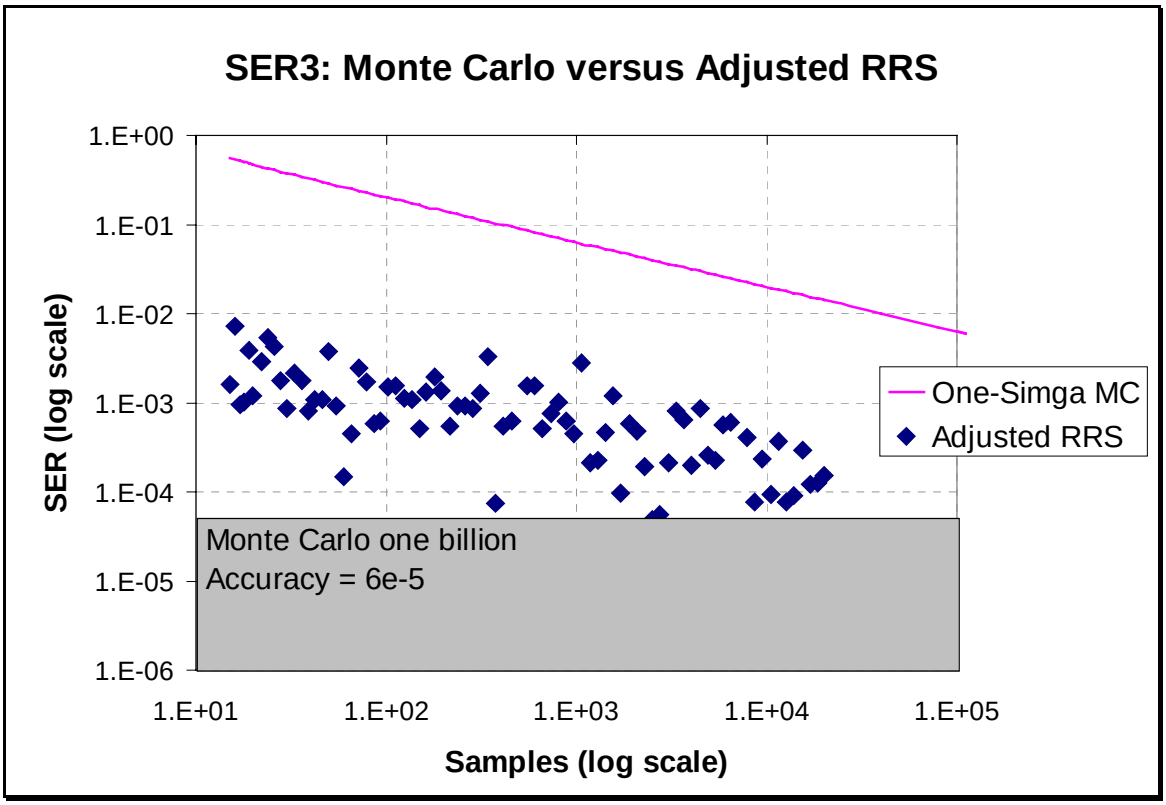


Figure 12-3: Comparing SER3 for Adjusted RSS and Monte Carlo

The accuracy of the adjusted RRS method will overtake the accuracy of a billion samples of Monte Carlo at about one hundred thousand samples, a factor of ten thousand fewer samples. The accuracy in skewness for the adjusted RRS method does not level off as with the RRS method alone. Figure 12-4 below shows the adjusted RRS results for SER4.

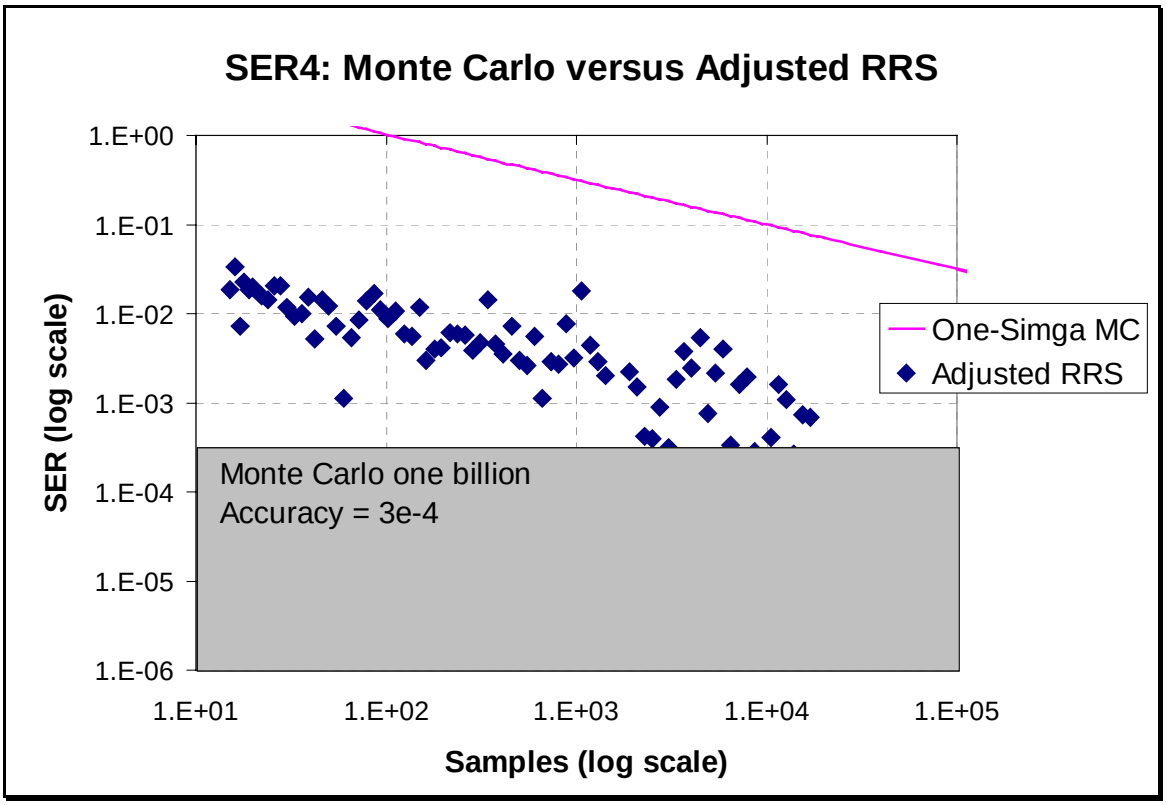


Figure 12-4: Comparing SER4 for Adjusted RSS and Monte Carlo

The accuracy of the adjusted RRS method will overtake the accuracy of one billion samples of Monte Carlo at about one hundred thousand, a factor of ten thousand fewer samples. Adjusting the moments from the RRS method with the estimates from the MCD method definitely increases the accuracy of skewness and kurtosis.

12.3.3. Total Error for the Random Response Surface Method

It has been shown that the error for fitting a real function to a quadratic can be estimated. Additionally, that very error can then be used to achieve an even better estimate of the output distribution moments. The major difficulty in using the RRS method is determining the final error after adjustment. At this point it can not be assumed that all assembly functions will follow the patterns seen for the clutch assembly. Because the

RRS method uses Monte Carlo, and the estimate of error from Monte Carlo, the RRS method can not be less accurate than Monte Carlo alone.

Section 12.4. Summary of the Random Response Surface Method

The RRS method is a general method of using a response surface generated from random points to determine the sensitivities to be used for the method of system moments. The method of system moments will estimate the first four output distribution moments, and then the Lambda distribution (or another general distribution) can be used to determine PPM rejects or other analysis objective functions.

The random points (generated like a Monte Carlo simulation) are important for two reasons. First, the entire design space will be explored according to the probability distribution. Thus, the response surface will represent the best fit of the function in the regions that it will occur naturally. Skewed input variables, for example, will not only affect the MSM during the estimation of the output distribution moments, but they will also affect where the emphasis is placed for fitting the response surface to obtain the sensitivities.

The second reason that the random points are important is for a good estimate of the standard moment errors. Because the points are generated with Monte Carlo, the estimate for the moments from the data and the fit equation will be representative of the true moments. These error estimates can then be used to adjust the output distribution moments to increase the accuracy of the overall analysis. Because the assembly problem chosen is one that is known to be quite non-quadratic, the accuracy of the RRS method for the clutch problem should be a conservative estimate for most other problems.

One very important feature of the RRS method is that it generates very accurate estimates of the quadratic sensitivities, not just the linear sensitivities like the correlation coefficients presented in Section 3.4. These quadratic sensitivities are used by the method of system moments, but may also be used to change the design and minimize the

sensitivities. Additionally, the relative accuracy of the sensitivities can be determined by looking at the standard moment errors estimated from the fit equation. It is recommended that further investigation into determining the accuracy of the sensitivities be conducted. It is also recommended that the RRS method be tested on a variety of sample problems to determine what factors affect the accuracy (non-quadratic nature of the assembly function, round-off error from calculations, etc.)

Chapter 13. Matching the Error in the Problem with the Analysis Method

Albert Einstein said, “Things should be made as simple as possible, but not any simpler.” The complexity and amount of calculation for variation analysis increases with the desired level of accuracy. Larger sample sizes increase the accuracy of Monte Carlo. A three-level DOE increases the accuracy over a two-level design. The quadratic estimate of the method of system moments is more accurate than a linear analysis, but it requires more sensitivities, input variable information, and many more calculations. The RRS hybrid method presented in Chapter 12 is extremely accurate, but requires precise input variable information, random generation of points for the response surface, second-order sensitivities for the MSM, and a fitted Lambda distribution if the PPM rejects are desired.

Additional accuracy in the analysis can not compensate for errors in the problem input information. Thus, the accuracy of the analysis method should be matched to the error level defined in the problem statement. Additional accuracy in the analysis would increase the amount of calculations, the complexity of the solution, and the difficulty in presenting or selling the results. If a greater level of accuracy is required, the input information must be improved first.

This chapter will investigate how inaccuracies in input information affect the accuracy of the solution, and how to match the accuracy of that information with an analysis method. The effect of inaccuracies in the input information is dependent on the analysis objective function: PPM rejects, quality loss, or sensitivities and percent contribution. The general methodology for estimating the effects of the errors in the input information is shown in Figure 13-1 below.

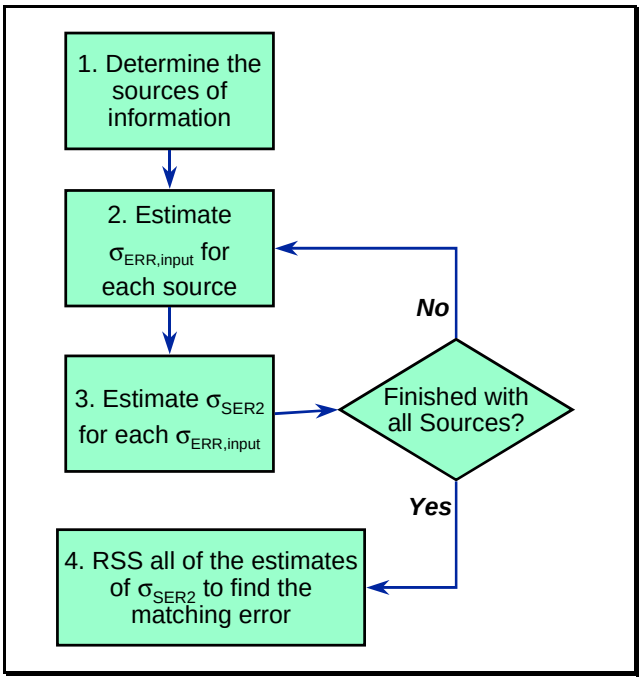


Figure 13-1: General Flowchart for Evaluating the Error in the Problem Information

Steps two and three must be performed for each piece of input information. All of the estimates of σ_{SER2} must be summed as the root-sum-square because they are independent. The estimate of SER2 is used since most of the analysis objectives are most sensitive to the variance, and the other standard moment errors will generally be proportional.

1. Each piece of input information should be identified, and even the source of the information determined where possible.

2. For each piece of input information, the uncertainty should be determined as a standard deviation of error (σ_{ERR}). For example, what is the 68 percent confidence interval for one of the specification limits (plus and minus one sigma error)?

3. The effect of each σ_{ERR} on the analysis objective function should be estimated and equated with a σ_{SER2} that results in the same level of uncertainty. In other words,

what level of σ_{SER2} would result in the same level of uncertainty for the objective function as the σ_{ERR} for that source of input information.

4. After all of the estimates of σ_{SER2} are found they can be summed as the root-sum-square to estimate the total one-sigma bound on SER2 from the input information. This bound on SER2 will be used to match the appropriate analysis method.

If the one-sigma bound on SER2 does not provide the accuracy that is required from the analysis, then the sources of error can be reduced by starting with the largest of the contributors (found in step 3). Once the one-sigma bound on SER2 is small enough, the analysis method can then be matched.

Section 13.1. Accuracy of Input Information versus PPM Rejects

Determining PPM rejects requires several types of input information: the assembly function, input variable information, and the specification limits. The most difficult factor affecting inaccuracies is the assembly function.

13.1.1. Error from an Incomplete Assembly Function

If some of the possible sources of variation in the assembly are eliminated from the model for simplification, then the magnitude of those variations should be estimated, and compared with the total variation to estimate the standard moment error for variance expected from the model simplification. That one-sigma bound on the standard moment error in variance can then be added to the other estimates of σ_{SER2} to match the appropriately accurate analysis method.

13.1.2. Error from Input Variable Information

What constitutes input variable error? Moments of measured data describe the input variables, and are typically estimated from a sample. The difference between the

estimated moments $\hat{\mu}_i$ and the actual moments μ_i is the input moment error. Dividing by the proper power of the input variable standard deviation transforms the errors into standard moment errors. Eq. 13-1 shows the standard moment error for variance for an input variable.

$$SER2_i = \frac{\hat{\mu}_{2,i} - \mu_{2,i}}{\sigma_i^2} \qquad \textbf{Eq. 13-1}$$

The actual errors for the input variables are not known, but the one-sigma bound on those errors (σ_{SER}) is easily estimated from the sample size. First the relationship between SER2 for the output and the input variables will be presented, and then the relationship between the one-sigma bounds on error.

Section 10.1 showed that the SER of the output distribution moments is very similar to the magnitude of the SER for the input variable moments (at least for the variables which contribute the most). For an exactly linear model, the SER2 for the output distribution is equal to the sum of the percent contribution of each input variable times its SER2. The derivation is shown below in Eq. 13-2.

$$\text{Output Variance} = \mu_2 = \sum_{i=1}^{n} S_i^2 \sigma_i^2 \qquad \textbf{Eq. 13-2}$$

$$SER2 = \frac{\hat{\mu}_2 - \mu_2}{\mu_2} = \frac{\sum_{i=1}^{n} S_i^2 \hat{\sigma}_i^2 - \sum_{i=1}^{n} S_i^2 \sigma_i^2}{\mu_2}$$

$$= \sum_{i=1}^{n} \frac{S_i^2}{\mu_2}\left(\hat{\sigma}_i^2 - \sigma_i^2\right) = \sum_{i=1}^{n} \frac{S_i^2 \sigma_i^2}{\mu_2} \frac{\left(\hat{\sigma}_i^2 - \sigma_i^2\right)}{\sigma_i^2}$$

$$= \sum_{i=1}^{n} \left(\%\text{Contribution}_i\right)\left(SER2_i\right)$$

Where: S_i = The linear sensitivity of the ith input variable

$\hat{\sigma}_i$ = The estimate of σ_i (the standard deviation of the ith input variable)

$\hat{\mu}_2$ = The estimate of μ_2 (the variance of the output distribution)

Because SER2 for the output distribution is a simple linear function of SER2 for each of the input variables, the σ_{SER2} for the output can be estimated as a linear variation problem (RSS). Eq. 13-3 below shows how the error estimates add by RSS.

$$\sigma_{SER2} = \sqrt{\sum_{i=1}^{n} \left(\%\text{Contribution}_i \sigma_{SER2,i}\right)^2} \qquad \textbf{Eq. 13-3}$$

Therefore, the estimate of the total σ_{SER2} for the output distribution can be found from the one-sigma bound for the variance of the input variables. For example if the σ_{SER2} for the input variables for the clutch assembly were all about 0.03 (estimated from about 2,000 samples), and one variable contributed most of the variation, then the total σ_{SER2} for the output distribution would also be about 0.03. Thus running a Monte Carlo simulation greater than 2,000 samples to determine the output distribution moments or PPM rejects would not produce much more accuracy.

13.1.3. Error from Uncertain Specification Limits

Specification limits are normally determined by the functional requirements of the assembly. Those assemblies that fall outside the specification limit are counted as rejects because they do not function correctly. But often, setting those specification limits involves safety factors, and guesswork. To determine how an error in determining the specification limit affects SER2 for the output distribution, the effect of the variance is shown below in Eq. 13-1.

$$\sigma_{output} = \frac{X_L - \overline{X}}{Z} \qquad \textbf{Eq. 13-1}$$

$$\Delta\left(\sigma^2_{output}\right) = \frac{2\left(X_L - \overline{X}\right)}{Z^2}\Delta X_L = \frac{2\sigma_{output}}{Z}\Delta X_L$$

Where: Z = The sigma level of the specification limit

ΔX_L = The error in the value of the specification limit (X_L)

$\Delta\left(\sigma^2_{output}\right)$ = The required change in variance for the distribution, corresponding to a change in the value of the specification limit, if the quality level is held constant

Thus, the change of variance to maintain a constant sigma quality level is proportional to the change in the limit (assuming relatively small changes). When that change in variance is incorporated into the definition for SER2, the expression can be simplified, as shown by Eq. 13-3.

$$SER2 = \frac{\left(\sigma^2_{output} + \frac{2\sigma_{output}}{Z}\Delta X_L\right) - \sigma^2_{output}}{\sigma^2} = \frac{2\Delta X_L}{Z\sigma_{output}} \qquad \textbf{Eq. 13-2}$$

$$SER2 = \left(\frac{2}{X_L - \overline{X}}\right)\Delta X_L$$

SER2 for the output is a simple linear function of the error in the specification limit. But the exact error in the specification limit is not known. So σ_X (or the one-sigma bound on the value of the specification limit) will be used to estimate σ_{SER2} for the output, as shown below in Eq. 13-3.

$$\sigma_{SER2} = \frac{2\sigma_X}{X_L - \overline{X}} \qquad \textbf{Eq. 13-3}$$

σ_{SER2} for the output distribution (the one-sigma bound on the value of SER2) is proportional to the error in the value of the specification limit divided by the distance of the specification limit to the mean. This approximation is good when the errors in the specification limits are relatively small compared to the standard deviation of the output distribution. In the example of the clutch assembly, the upper specification is set at 7.6184, and the mean is about 7.0184 (nominal). If the uncertainty (68 percent confidence interval) were just the last significant digit (0.0001), then σ_{SER2} = 2(0.0001)/(7.6184-7.0184) = 0.000333. That is the quality level achieved by 18 million samples of Monte Carlo (see Eq. 9-2) or about 1,000 samples of the adjusted RRS hybrid method (see Figure 12-2).

Section 13.2. Accuracy of Input Information versus the Quality Loss Function

The Taguchi quality loss function was presented in Section 6.1 and the sensitivities with respect to the standard moment error were presented in Section 8.3. The extra information required for the Taguchi quality loss function (not required to estimate PPM rejects) is the cost constant. Assuming that the difference between the mean and the minimum cost point is small relative to the standard deviation of the output distribution, Eq. 13-1 shows the estimation for SER2 versus the error in estimating the cost constant (holding the quality loss as a constant).

$$L = K\sigma^2 \qquad \textbf{Eq. 13-1}$$

$$\Delta\sigma^2 = -\frac{L}{K^2}\Delta K = -\frac{\overset{\sigma^2}{L}}{K}\frac{1}{K}\Delta K = -\frac{\sigma^2}{K}\Delta K$$

$$SER2 = \frac{\Delta\sigma^2}{\sigma^2} = -\frac{\Delta K}{K}$$

SER2 for the output is again a simple linear function of the error in the cost constant. The exact error in the cost constant is not known. But the one-sigma bound on the error of K (or σ_K) can be used to determine the corresponding σ_{SER2} for the output distribution.

$$\sigma_{SER2} = \frac{1}{K}\sigma_K \qquad \textbf{Eq. 13-2}$$

The error from estimating the minimum cost point can be included into the uncertainty of the cost constant K. The cost constant is based on an estimate for the minimum cost point. The cost constant for the clutch assembly is based on an estimate that it costs \$20 (lost in scrap) if the contact angle is 0.6 degrees off the optimum. If the uncertainty is about \$1 in the scrap cost (depending on batch size, etc.) then the uncertainty in the cost constant would be \$2.78, and the corresponding one-sigma bound on SER2 (σ_{SER2}) would be 0.05. This represents the accuracy of only 800 samples of Monte Carlo.

13.2.1. Estimating the Total σ_{SER2} for the Input Information

In the case of estimating the PPM rejects for the clutch assembly, the sources of informational error were the input variable moments (SER2) and the specification limits. Hypothetical σ_{SER2} estimates for the different sources were presented, and now they will be combined. Eq. 13-1 below shows how the two main sources of input information error affect the overall accuracy.

$$\sigma_{SER2,total} = \sqrt{\left(\sigma_{SER2,\text{variable moments}}\right)^2 + \left(\sigma_{SER2,\text{specification limits}}\right)^2} \qquad \textbf{Eq. 13-1}$$

$$\sigma_{SER2,total} = \sqrt{(0.03)^2 + 2(0.000333)^2} = \sqrt{9.0\times10^{-4} + 2.2\times10^{-7}}$$

$$\sigma_{SER2,total} = 0.03$$

It is easy to see from the calculation above that the error in estimating of the variances of the input variables is contributing virtually all of the error in the problem. The error term for the specification limits is multiplied by two because there are two limits (an upper and a lower). The reason that the uncertainty in the cost constant was not included is that the estimate was made assuming PPM rejects was the analysis objective function. If a more accurate estimate of rejects is desired than the $\sigma_{SER2,total}$ can provide, the input variable distributions should be determined more accurately.

In the case of quality loss the main sources of informational error are the input variables and the cost constant (K). The calculation of the total σ_{SER2} due to the uncertainty of the parts is shown below in Eq. 13-2.

$$\sigma_{SER2,total} = \sqrt{\left(\sigma_{SER2,\text{variable moments}}\right)^2 + \left(\sigma_{SER2,K}\right)^2} \qquad \textbf{Eq. 13-2}$$

$$\sigma_{SER2,total} = \sqrt{(0.03)^2 + (0.05)^2} = \sqrt{9\times10^{-4} + 2.5\times10^{-3}}$$

$$\sigma_{SER2,total} = 0.058$$

In the hypothetical case of quality loss, the major contributor to $\sigma_{SER2,total}$ is the error in estimating the quality loss constant, in fact 96 percent of $(\sigma_{SER2,total})^2$. Often the determination of the cost constant is difficult, and therefore done with rough estimation.

Section 13.3. Matching the Errors of the Analysis Methods

Once the $\sigma_{SER2,total}$ (one-sigma error bound on the standard moment error of variance) has been determined the analysis method can be chosen to match the error. Having calculated the error in terms of σ_{SER2}, it is relatively easy to match the accuracy for the methods discussed.

13.3.1. Matching Error with Monte Carlo

The σ_{SER2} for Monte Carlo is easily determined based on the sample size. Eq. 13-1 below shows how the sample size can be determined according to the desired level of confidence for SER2.

$$\sigma_{SER2} = \sqrt{\frac{2}{n-1}} \qquad \textbf{Eq. 13-1}$$

$$n = \frac{2}{\sigma_{SER2}^2} + 1$$

The equation above is probably not very good for small sample sizes (less than about 100). But, for the hypothetical clutch problem, if the PPM rejects were desired, Eq. 13-1 estimates that $2/0.03^2 + 1$, or about 2,200 (the +1 is insignificant) samples would be required for Monte Carlo Simulation.

13.3.2. Matching Error with Linear Variation Analysis

Table 10-1 in Section 10.2 showed how the quadratic ratio is estimated for the input variables. The hub radius was the major contributor to variance, and also has the largest standard deviation of the input variables. Therefore, the quadratic ratio for the hub radius is a good estimate for the linearization error.

The linearization error predicted by the hub radius ***a*** for SER2, or SER2a, was –0.0004. Comparing that to the $\sigma_{SER2,total}$ for the PPM rejects (0.01) or the quality loss (0.051)

suggests that a linear analysis is accurate enough. While SER2a, predicted from the hub radius quadratic ratio, is not an exact approximation for the real error, it does indicate that using a quadratic model instead of a linear model would not be valuable in this case. The clutch assembly was chosen because it is significantly non-quadratic, therefore the quadratic ratio should be an even better predictor for most assembly problems.

13.3.3. Matching Error with Design of Experiments

In this case, matching the error with DOE is very similar to evaluating the linear analysis. SER2a, predicted by the hub radius quadratic ratio, indicates that a two level design is accurate enough. The regular two and three level designs are not designed to predict the skewness and kurtosis of the output distribution, but if the input variables are normally distributed, the output can be assumed to be normally distributed as well with the two-level design.

If the estimate of the SER2, from the quadratic ratio, were borderline with $\sigma_{SER2,total}$, that single most quadratic variable could evaluated with three levels, while the others could remain with two levels. If the SER2 estimated by the quadratic ratio were extremely small (assembly function almost entirely linear), then even an orthogonal design could be safely used.

13.3.4. Matching Error with the Random Response Surface Method

The random response surface hybrid method is very accurate at small sample sizes relative to Monte Carlo. So, in the few cases when input information is very accurate, and the desired accuracy of the analysis very high, the RRS method should be used. Not only will the accuracy of the RRS method exceed Monte Carlo, but it will also yield a set of accurate quadratic sensitivities. Sometimes the RRS method could be the optimal choice not because of its accuracy, but because of the sensitivities and the quadratic response surface generated. Additionally, the RRS method is well suited for the input variables which have general probability distributions (not necessarily normal or symmetrical).

Chapter 14. Choosing between the Analysis Methods

This chapter will first directly compare many aspects of the different variation analysis methods, and then it will present a table to aid in selecting the correct method for the analysis problem.

Section 14.1. Direct Comparison of the Methods

The assumptions and limitations, required inputs, outputs, accuracy, and the method specialties must be compared in order to choose between the analysis methods. The different analysis methods to be evaluated are Monte Carlo (**MC**), method of system moments (**MSM**), first-order approximation (**RSS**), design of experiments (**DOE**), response surface method (**RS**), and the random response surface hybrid method (**RRS**).

14.1.1. Method Assumptions and Limitations

- **MC**: Because of the extremely large sample sizes required, the assembly function should be represented in mathematical form (explicit equation preferred over implicit), and evaluated through computer simulation.

- **MSM**: The assembly function is assumed to be well represented by a quadratic function. Because of the extremely large number of terms required to calculate the full quadratic estimates for skewness and kurtosis, the equations should be evaluated using a computer.

- **RSS**: The output distribution is generally assumed to be normal. Assemblies with small numbers of parts should have normally distributed input variables. The assembly function is assumed to be well represented by a linear function. The mean of the output distribution is assumed to the value of the assembly function at the nominal values for the input variables.

- **DOE**: The two-level designs assume the assembly function is linear plus the interaction terms. The three-level designs assume the assembly function is quadratic. The input distribution are assumed to symmetric and uniform (normal for the three-level weighted design).

- **RS**: Solving the linear algebra equation for the coefficients of the response surface requires inverting a matrix. Because of the number of coefficients, problems of more than three variables would only be practical with the aid of the computer.

- **RRS**: The random response surface method has all of the restrictions of MC, MSM, and RS. Although RRS is more tolerant to implicit equations as the required sample size is much smaller than MC.

14.1.2. Required Inputs

- **MC**: The probability distributions for the input variables are required to generate the random points. A random number generator must be used appropriate to the sample size.

- **MSM**: The first eight moments of the input variables should be known. The first and second-order sensitivities must also be known.

- **RSS**: Only the mean and standard deviations of the input variables are required. The first-order sensitivities are also required.

- **DOE**: The mean and standard deviations of the input variables are required.

- **RS**: The design space (upper and lower practical limits of the input variables) is used to generate the response surface design points. If the response surface is to be used to generate the sensitivities for MSM, the input variables should be samples at about 1.4σ.

- **RRS**: The random response surface method requires the same inputs as MC, the input variable distributions and random numbers.

14.1.3. Outputs

- **MC**: The output distribution moments can be estimated, along with a direct estimation of the PPM rejects. An estimation of the first-order sensitivities can be found from the correlation coefficients.

- **MSM**: The first four output distribution moments are estimated. The PPM rejects can be found by fitting a Lambda (or other) distribution to the four output distribution moments.

- **RSS**: The variance of the output distribution is estimated. The skewness and kurtosis can be estimated, but are often assumed to be normal. A normal distribution can be fitted to the nominal output and the estimate of variance to obtain an estimate for PPM rejects.

- **DOE**: The mean and variance of the output distribution are determined. The sensitivities and contribution to variance of the different factors are easily determined. The three-level weighted design can estimate the skewness and kurtosis of the output distribution as well. A normal or Lambda distribution can then be fitted to the data and used to estimate PPM rejects.

- **RS**: An approximation (often quadratic or linear) of the assembly function is determined which best models the variance of the points sampled. The first and second-order sensitivities can be easily determined from the response surface as a function of the input variables.

- **RRS**: The random response surface method estimates the four output distribution moments, the response surface that best models the variance of the actual output

distribution, and the first and second-order sensitivities. The PPM rejects can be determined by counting the rejects during the Monte Carlo portion of the method, or a Lambda distribution can be fit to the output distribution to estimate the PPM rejects at any value of the specification limits (certainty limited by the Lambda distribution fit).

14.1.4. Accuracy

- **MC**: The accuracy of counting rejects is inversely proportional to the square root of n-1 (Eq. 3-1). The accuracy of the standard moment errors is proportional to the square root of n for SER1 (Eq. 9-1), n-1 for SER2 (Eq. 9-2), n-2 for SER3 (Eq. 9-3), and n-6 for SER4 (Eq. 9-4). The accuracy of estimating rejects from the moments, instead of counting, is virtually the same as for counting, but limited by the accuracy of the fit distribution (Eq. 9-1 and Figure 9-1). The accuracy of using Monte Carlo to estimate quality loss is approximately proportional to the square root of n (Eq. 9-1).

- **MSM**: The accuracy of the method of system moments depends on the accuracy of the sensitivities, and the how non-quadratic the assembly function is. Figure 10-1 showed that MSM is much more accurate when using sensitivities from a response surface sampling at about 1.4 times the standard deviations of the input variables than using the exact or finite-difference derivatives. The Monte Carlo Difference method (Section 10.1.2) can be used to estimate SER1–4 for MSM at much fewer samples than required with Monte Carlo alone.

- **RSS**: The quadratic ratio, QR = $\frac{\frac{1}{2}\frac{\partial^2 f}{\partial a^2}\sigma_a}{\left|\frac{\partial f}{\partial a}\right|}$, is a dimensionless ratio useful in estimating the linearization error. The quadratic ratio estimates the error SER1 for assuming a quadratic function is only linear (Eq. 10-2). The errors SER2 through SER4 also can be estimated from the quadratic ratio (Eq. 10-5). If the input variables

are normally distributed, the estimates of SER1 through SER4 are simplified by Eq. 10-6.

- **DOE**: The quadratic ratio (see above) can also be used to estimate the error of using only two levels instead of three for a single input variable. Thus, only the most non-linear variables must use three levels, while the rest can use two. Thus, the accuracy can be significantly enhanced while not increasing the number of runs dramatically.

- **RS**: The accuracy of a response surface depends on how well the fitted response surface compares to the shape of the actual function being fit. The coefficient of determination, Eq. 12-8, determines the percent of the raw variation in the data that can be explained by the fit equation.

- **RRS**: The accuracy of the random response surface method depends on how non-quadratic the assembly function is. Estimates for SER2–4 between RRS and MC are obtained by the Monte Carlo Difference method, Eq. 12-1, and can be used to adjust the output distribution moments to obtain even more accuracy. But the accuracy can not be worse than for Monte Carlo method, and with the clutch assembly it is 100 times more accurate ($SER_{RRS} = SER_{MC}/100$). The sensitivities are very accurate, as they are obtained by sampling the probable design space, not necessarily symmetrical.

14.1.5. Method Specialties

- **MC**: Monte Carlo is very versatile. It can be used with any type of assembly function (non-continuous, implicit, sinusoidal, etc.), and any type of input variable distributions (as long as the random deviate can be generated). The input variable variations can even be correlated. Monte Carlo can also be used to estimate the higher moments (more than just the first four moments).

- **MSM**: The method of system moments is useful in that the input information can be changed and the effect on the output distribution determined instantly. If the variation

of an input variable is reduced, not only will it reduce the output variance, but it will also estimate the new value of skewness and other moments of the output distribution.

- **RSS**: Linear RSS analysis is useful, as only the first-order sensitivities and the input variable means and variances are required. It is easily understood and used. Even more than with MSM, the input information can be altered, and the effect determined instantly. In fact, the output distribution mean and variance can be specified as a design requirement, and the input information necessary to reach it may be determined. Linear explicitization (Section 5.3) is a method used with RSS that takes an implicit equation and easily converts it to an explicit linear equation, ready for the RSS method.

- **DOE**: DOE is the only method that can work even if there is no mathematical expression or sensitivities for the assembly model, or if all of the sources of variation to be examined are not understood. Physically building the parts and testing them can even help determine the sources of variation. DOE is especially good at breaking up the total variance in the output into its component pieces.

- **RS**: A response surface is especially good at turning a complex assembly function into one that is more easily studied (quadratic, linear, etc.) The sensitivities of the output function can be determined as functions of the input variables, and thus the design can be optimized to reduce the sensitivities.

- **RRS**: The random response surface method is extremely accurate at determining the output distribution moments, and the first and second-order sensitivities like the response surface method.

Section 14.2. Table of Methods

Bringing all of the methods together and comparing accuracy is extremely difficult as each method requires different inputs and provides different types of information about

the assembly function. But Table 14-1 below attempts to match the methods with the analysis objectives according to accuracy.

Table 14-1: Recommended Methods for Balancing Input Information and Analysis Accuracy

Accuracy	Required Input Variables	Function	Method for Desired Output PPM Rejects	Quality Loss	Sensitivities
High	**μ_{1-8} or Known Distribution**	**Explicit**	**MC***	MC*	RRS
		Implicit	**RRS**	RRS	RRS
		S_1 and S_2	**MSM**	MSM	
Medium	**Distribution Assumed Normal**	**Explicit**	DOE(mixed)	**DOE(mixed)**	RS / FD
		Implicit	DOE(mixed)	**DOE(mixed)**	RS / FD
		S_1 and S_2	MSM	**MSM**	
Low	**μ_{1-2}**	**Explicit**	**DOE2 / RSS**	**DOE2 / RSS**	**RS / CATS**
		Implicit	**RSS(CATS)**	**RSS(CATS)**	**RS / CATS**
		S_1	**RSS**	**RSS**	

** If sensitivity information is also desired the RRS method is preferred*

S_1 and S_2 refer to first and second order sensitivities, respectively. The analysis methods in bold are the ones where the accuracy and the analysis objective are a very practical combination. For example, if only the quality loss is desired (only a function of the mean and variance) high levels of accuracy for the input variable moments (particularly the skewness or kurtosis) are not very necessary, and only relative levels of loss are needed.

Table 14-1 is only for general problems (not known to be linear or quadratic) and general input distributions. Many different combinations and methods are not shown in the table above. For example the weighted three-level experimental design is not shown because it is extremely accurate if the input variables are normally distributed. Often input variable

distributions are assumed to be normal, and often are approximately normal. But, if a very high level of accuracy is desired, the input distribution can not be assumed.

Additionally, if the assembly function is known to be linear, then simple linear RSS should be used, regardless of the required analysis accuracy. And if linearity is suspected, the quadratic ratio Section 10.2.1 can help estimate the level of accuracy of using RSS or a two-level DOE, and thus prevent the need of using Monte Carlo or a three-level DOE.

Section 14.3. Examples of Method Selection over the Product Life Cycle

To conclude this chapter, the insights into the different methods and Table 14-1 will be used to determine appropriate methods to be used during a hypothetical product life cycle of the clutch assembly. Some of the times during the life cycle when variation analysis would be useful are the following:

- Product Design Stage
- Tolerance and Process Selection Stage
- Quality Improvement Stage

14.3.1. Variation Analysis in the Product Design Stage

In the product design stage, the exact dimensions for the parts in the assembly may not be known. In fact the number of parts in the assembly may not even be sure. The general processes to make most of the parts may be known. The required output from a variation analysis problem would probably be the sensitivities or just a rough estimate of assembly variation and percent contribution to variance. Probably not every source of variation for the assembly will be included or known.

In this situation, the accuracy of the input information is very low, and the required accuracy of the output is also low. A simple response surface fit would help determine the

sensitivities of the assembly as a function of the nominal values. Thus, the nominal part dimensions could be chosen to decrease the sensitivity of the assembly performance with respect to the parts with the largest sensitivity, or the parts most difficult to manufacture.

At most a simple RSS linear analysis or two-level DOE could be performed to estimate the overall variation and the percent contribution. Thus, in the early stages of product design, a response surface or a linear analysis would be valuable.

14.3.2. Variation Analysis in the Tolerance and Process Selection Stage

In this stage of the product life cycle, the assembly design is well defined dimensionally, and the types of processes selected. The variations corresponding to the selected processes are assigned to the respective parts (assumed normally distributed). The specifications for the assembly are probably known with more certainty than the variations of the parts. The role of variation analysis in this situation would be to determine if the variation in the assembly due to the variation in the parts causes a greater number of rejects or the quality loss to be greater than desired.

The level of accuracy for the input information is medium at this stage, greater than the previous stage. The variation analysis method that would be most useful would be a mixed DOE. The quadratic ratio could be used to determine the magnitude of the non-linearity in the assembly function. If the assembly function is linear, or very close to it, a two-level DOE or RSS method could be used. But, if one or more of the variables have large quadratic ratios relative to the desired level of accuracy (see Section 13.3.3) those few variables could be assigned three-levels of a mixed DOE.

Once the overall variation and percent contribution is determined, the quality loss or estimate of PPM rejects can be made. If the assembly variation is too great, the production processes for the largest contributors can be changed to reduce the overall variation. If the assembly variation is much smaller than required, the more expensive processes can be replaced by less-expensive processes. Thus, variation analysis in the

process selection stage can help to determine an optimum set of production processes to minimize the quality loss, while minimizing the production cost.

14.3.3. Variation Analysis in the Quality Improvement Stage

This stage normally occurs after the product has been produced for some time. The competitive pressures (on price and quality) are forcing the company to look for areas of improvement. Variation analysis can help find ways to either increase the quality at minimal cost, or to lower the cost while maintaining the quality. At this stage, changes in production are relatively expensive compared to earlier stages. Thus, a greater level of certainty is required for the variation analysis. But, greater information on the sources of variation, input variable distributions, function requirements, and the quality loss function are known because the parts have been in production for a while.

The variation analysis method most useful in this situation, assuming the assembly function is not linear, is the random response surface hybrid method. The confidence level of the results must be high, and the sensitivity information from the RRS method is also valuable. The RRS method would determine the optimum linear and quadratic sensitivities to examine the effect of slight modifications in the design (shifting the nominal dimensions). Effects of slight modifications in the input variable variances (machine overhaul or new tooling) can be evaluated instantly, rather than having to redo a Monte Carlo simulation.

If the potential changes are great (changing processes, materials, or even changing the design significantly) then the accuracy of the input information would probably be reduced. The variations of the input variables would probably be based on estimates of different processes rather than on historical data. Thus the variation analysis situation would be similar to the processes selection stage, except that the required level of certainty would be greater, as changes in production are more expensive at this point.

Section 14.4. Summary for Choosing Between the Analysis Methods

The framework created in this chapter helps to optimize the variation analysis process. Matching the accuracy of the analysis method to the accuracy of the input information prevents over-analyzing a problem and overstating the confidence of the results. Comparing the different method with the inputs and accuracy in Table 14-1 helps to use the right tool for the job. The variation analysis model should be simplified as much as the required accuracy allows.

Chapter 15. Conclusion and Recommendations for Further Research

Four methods for performing variation analysis of mechanical assemblies have been compared in great detail. The required input information and predicted assembly resultants have been reviewed, along with a systematic evaluation of their respective accuracy and efficiency. Six main assembly variation parameters of interest to designers and manufacturing personnel were used throughout the study: the first four moments of the resulting statistical distribution, the predicted rejects at the upper and lower design limits, and the Taguchi quality loss function. The capability of each variation analysis method was evaluated relative to those parameters.

The first part of the thesis (Chapter 2 through Chapter 6) demonstrated the state-of-the-art in variation analysis, and prepared the reader to understand and appreciate the contributions from the second part of this thesis (Chapter 8 through Chapter 14). These later chapters focused on the unique contributions made by this thesis. The main contributions of this thesis were:

- Developed a comprehensive procedure for evaluating variation analysis methods, based on a thorough error analysis of the effects of input errors, assembly function approximations, and statistical computation methods.

- Presented and demonstrated new metrics for estimating the accuracy of the different variation analysis methods based on an extension of the central limit theorem and the standard error of the mean to each of the six variation parameters (Chapter 8 through Chapter 12)

- Presented and evaluated a new hybrid method that combines the strengths of several different analysis methods (Chapter 9 through Chapter 11)

- Presented a comprehensive framework for choosing the method most suited for variation analysis based on the level of accuracy of the output parameters desired, the accuracy of the available data, the accuracy of the assembly function, the quality level, and the computational efficiency (Chapter 13 and Chapter 14)

Section 15.1. New Metrics: Standard Moment Error and Quadratic Ratio

The principle contribution of this thesis is the presentation and demonstration of new metrics for estimating the accuracy of the different variation analysis methods. Current methods for estimating the accuracy of variation analysis methods are not sufficient. The current methods include only comparing the results to a benchmark of Monte Carlo simulation.

The accuracy of counting rejects with Monte Carlo and estimating the error for the mean and variance are well defined in terms of the sample size, but often not used. Many times when comparing a method to Monte Carlo, the difference between the two methods is not statistically significant because of an inadequate sample size.

This thesis contributed new metrics: standard moment error and the quadratic ratio. The standard moment errors are non-dimensional measures of accuracy for the moments of a distribution. They are robust and are not a function of the actual magnitude of the error, unlike using percent error. Skewness error in particular is exaggerated when the actual value of skewness is near zero. The sensitivities of estimating PPM rejects and quality loss with respect to each of the standard moment errors are about the same order of magnitude.

The second significant metric is the quadratic ratio. This ratio is helpful in evaluating an individual dimension or independent variable. The ratio helps to estimate the magnitude of the error that would result in assuming that assembly function were only linear with respect to that variable. Therefore it is an invaluable metric to be used for estimating the accuracy of not only linear RSS, but also for refining the accuracy of design of

experiments. It was suggested that variables with large quadratic ratios can be modeled using three levels, while the others only need two. Thus, using a mixed design, the efficiency of the DOE is increased.

Section 15.2. New Hybrid Method: Random Response Surface (RRS)

From the presentation each of the standard variation analysis methods, it is clear that they each use different information and calculate different types of results. The output of the response surface method is the sensitivities for example, while the input for the method of system moments is the sensitivities. Thus, the methods can easily be combined to enhance their value.

The RRS hybrid method uses Monte Carlo to generate random points in the probable design space according to the input variable probability distributions. For each set of randomly generated component dimensions, the assembly function was used to calculate the critical assembly feature of design interest. A response surface is then fit to these points. The value of the fitted response surface at the same values of the inputs as the random points can be compared to the original function values to estimate the standard moment errors.

The method of system moments is then applied, using the sensitivities from the response surface, to estimate the output distribution moments. The estimates of standard moment error at this point are a rough estimate of the true error of the output distribution moments. In fact, the output moments from MSM can be adjusted by these estimates of error to further increase the accuracy of the method.

The result of the RRS hybrid method is a method that is much more accurate than Monte Carlo at the same number of samples (roughly one-hundredth the standard moment errors). Extremely good estimates of the sensitivities are also produced. And finally, there is an estimate of error corresponding to sample size to assist in selecting the appropriate sample size for the analysis.

Section 15.3. Framework for Choosing Accuracy and Method

The accuracy of the analysis should not exceed the accuracy of the input information. This is called matching the errors. This thesis has presented the idea that the most accurate method is not the best in most circumstances. An accurate analysis method can not correct for errors in the input variable distributions, for example.

Equations were presented to help determine how the uncertainty in the input information translates to uncertainty in SER2 (standard moment error for variance) for the output distribution. Thus, the simplest method that can achieve the same accuracy should be used (as long as it produces the desired types of output information). A table was created to help in making the selection. The level of accuracy and the types of output information were the main drivers of the table.

Section 15.4. Recommendations for Further Research

The main weakness of this thesis is that only one sample problem, the one-way clutch, was used throughout. It is thus recommended that the new metrics (SER and QR) be used on a variety of problems (linear, non-linear, non-quadratic, explicit, implicit, etc.), accompanied by their applications in estimating error and the system for choosing the appropriate analysis methods. Only after this type of verification can the value of the new metrics really be determined.

A method to combine the quadratic ratios from several variables to estimate their total effect (perhaps using their percent contribution) should be determined to better estimate the linearization error when one variable does not contribute most of the variation. Maybe even the SER1-4 errors can be used to adjust the input variable moments so that a linear analysis could be performed with the accuracy similar to MSM.

The optimal sampling ratios (for using a response surface to obtain the sensitivities for MSM) should be determined by evaluating many different assembly problems. Is the

optimum sampling ratio dependent on the assembly function, the number of variables, etc.

Additionally, the RRS hybrid method should be further tested on a variety of problems to verify its accuracy relative to Monte Carlo for the same number of samples. Perhaps a new ratio like the quadratic ratio (but for third-order effects) could be developed for this purpose. The Monte Carlo difference method (MCD) should also be tested on a variety of problems to determine the required number of runs to estimate the standard moment errors for the MSM and RRS methods.

The accuracy of the Lambda distribution limits the accuracy for estimating PPM rejects using the output distribution moments. A better method of fitting the Lambda distribution to the first four moments (depending how far out in the tails the distribution will be used) would be valuable to increase the accuracy when estimating PPM rejects.

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